

# Some Variants of Weak Pigeonhole Principle and WWKL

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# Motivation

## Theorem

- ▶ (Avigad, Dean and Rute)  $\text{RCA}_0 \vdash 2\text{-WWKL}_0 \rightarrow 2\text{-RAND}$ , where  $2\text{-RAND}$  asserts the existence of a 2-random real and  $2\text{-WWKL}_0$  is the statement that  $[T] \neq \emptyset$  for every  $\Sigma_2^0$  binary tree  $T$  s.t.

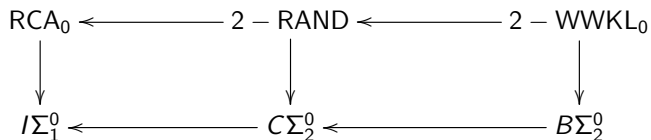
$$\exists \delta > 0 \forall n (|T \cap 2^n| > \delta 2^n).$$

- ▶ (ADR)  $\text{RCA}_0 \vdash 2\text{-WWKL}_0 \rightarrow B\Sigma_2^0$ .
- ▶ (Conidis and Slaman)  $2\text{-WWKL}_0$  is  $\Pi_1^1$ -conservative over  $\text{RCA}_0 + B\Sigma_2^0$ .
- ▶ (CS)  $\text{RCA}_0 \vdash 2\text{-RAND} \rightarrow C\Sigma_2^0$ , where  $C\Sigma_2^0$  states that every  $\Sigma_2^0$  injection on all natural numbers has an unbounded range.

# Motivation

## The Diagram

The arrows below are based on  $RCA_0$ :



# An Obstacle

## Theorem (Kučera)

*If  $T$  is a binary tree with  $\mu([T]) > 0$  and  $X$  is  $T$ -random then  $X$  computes some  $Y \in [T]$ .*

As 2-random reals are  $\emptyset'$ -random, Kučera's Theorem implies that **no** models like  $(\mathbb{N}, \mathcal{S})$  can separate  $2 - \text{WWKL}_0$  and  $2 - \text{RAND}$ .

## Theorem (Slaman, unpublished)

$\text{RCA}_0 + 2 - \text{RAND} \not\vdash 2 - \text{WWKL}_0$  *witnessed by non-standard models.*

# An Interpolation

$2 - \text{WWKL}_0(1/2)$ :  $[T] \neq \emptyset$  for every  $\Sigma_2^0$  binary tree  $T$  s.t.

$$\forall n (|T \cap 2^n| > 2^{n-1}).$$

If  $\mathcal{M} = (\mathbb{N}, \mathcal{S}) \models \text{RCA}_0$  then

$$\mathcal{M} \models 2 - \text{WWKL}_0 \Leftrightarrow \mathcal{M} \models 2 - \text{WWKL}_0(1/2).$$

## Theorem (BCWWY)

$\text{RCA}_0 \vdash 2 - \text{WWKL}_0 \rightarrow 2 - \text{WWKL}_0(1/2) \rightarrow 2 - \text{RAND}$ .

# Another Interpolation

## Theorem (Dimitracopoulos and Paris)

Over  $I\Sigma_n$  ( $n > 0$ ),  $B\Sigma_{n+1}$  is equivalent to that there is **no**  $\Sigma_{n+1}$  injection from any  $x + 1$  to  $x$ .

**$\Sigma_n$  - WPHP**: There is **no**  $\Sigma_{n+1}$  injection from any  $2x$  to  $x > 0$ .

Obviously,

$$C\Sigma_{n+1} \leftarrow \Sigma_{n+1} - \text{WPHP} \leftarrow B\Sigma_{n+1}.$$

# The First Order Theory of $2 - WWKL_0(1/2)$

## Theorem (BCWWY)

1.  $RCA_0 \vdash 2 - WWKL_0(1/2) \rightarrow \Sigma_2^0 - WPHP$ .
2. *Every countable  $\mathcal{M} \models RCA_0 + \Sigma_2^0 - WPHP$  can be expanded to a model of  $RCA_0 + 2 - WWKL_0(1/2)$ , so  $2 - WWKL_0(1/2)$  is  $\Pi_1^1$ -conservative over  $RCA_0 + \Sigma_2^0 - WPHP$  and the first order theory of  $2 - WWKL_0(1/2)$  is axiomatized by  $I\Sigma_1 + \Sigma_2 - WPHP$ .*

The second result is proved by forcing over a countable model  $\mathcal{M}$  with conditions  $(\sigma, S)$ , where  $\sigma$  is a finite (in  $\mathcal{M}$ ) binary sequence and  $S$  is a  $\Sigma_2^0$  binary tree with  $\mu([S]) > 1/2$ . The proof uses a restriction of Lebesgue's Density Theorem.

## $\Sigma_{n+1}$ - WPHP and $B\Sigma_{n+1}$

### Theorem (BCWWY)

$I\Sigma_n + \Sigma_{n+1}$  - WPHP  $\not\vdash B\Sigma_{n+1}$  ( $n > 0$ ).

The theorem depends on two technical results:

1. (BCWWY, Yokoyama) Let  $M \models I\Sigma_n$  and  $I$  be a  $\Delta_{n+1}$  proper cut in  $M$ . If  $M \prec_{n+1} N$  and  $N$  has **no** element between  $I$  and  $M - I$  then  $I$  is a  $\Delta_{n+1}$  proper cut in  $N$ .
2. (BCWWY) Let  $M \models I\Sigma_n$  and  $I$  be a  $\Delta_{n+1}$  proper cut in  $M$ . If  $a \in M$  and  $F : 2a \rightarrow a$  is a  $\Sigma_{n+1}(M)$ -injection then there exists  $N$  s.t.  $M \prec_{n+1} N \models I\Sigma_n$ ,  $N$  has **no** element between  $I$  and  $M - I$ , and  $N$  contains  $c < 2a$  with  $F(c)^N$  undefined.

The proof of the 2nd item above uses Paris' ultrapower construction: first carefully (!!!) constructs an ultrafilter  $U$  on  $M$ -finite subsets of  $2a$  and then takes

$$N = M - \text{finite functions } f : 2a \rightarrow M/U.$$



# $C\Sigma_{n+1}$ and $\Sigma_{n+1} - \text{WPHP}$

## Theorem (BCWWY)

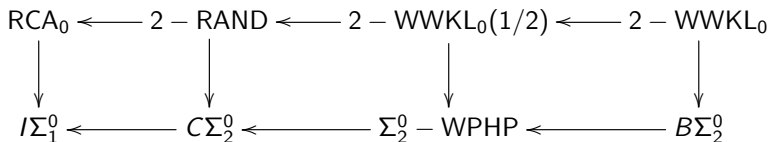
$I\Sigma_n + C\Sigma_{n+1} \not\vdash \Sigma_{n+1} - \text{WPHP}$  ( $n > 0$ ).

There are two proofs of the above theorem:

1. The 1st uses a so-called flexible formula constructed by Blanck and Enayat and obtains  $N = \bigcup_k N_k \models C\Sigma_{n+1} + \neg\Sigma_{n+1} - \text{WPHP}$ , where  $N_k \prec_{n+1,e} N_{k+1} \models \text{PA}$ . This can also be applied to get  $I\Sigma_1 + \bigcup_{k \in \mathbb{N}} C\Sigma_k + \neg B\Sigma_2$ , an unpublished theorem of Slaman by a similar proof.
2. The 2nd uses Paris' construction again to obtain a technical lemma: for countable  $M \models I\Sigma_n$  and  $1 < a \in M$  and a  $\Sigma_{n+1}(M)$ -injection  $F : M \rightarrow a$ , there is  $N$  s.t.  $M \prec_{n+1,cf} N \models I\Sigma_n$ ,  $[0, a]^M = [0, a]^N$  and  $F^N$  is partial; then it builds a chain of cofinal extensions beginning with a projection model of  $I\Sigma_n$  (i.e., some  $M \models I\Sigma_n$  having a  $\Sigma_{n+1}$  injection  $P : M \rightarrow \mathbb{N}$ ).

# A New Diagram

The arrows below are based on  $RCA_0$ :



## ... and a Question

Is the first order theory of  $2 - \text{RAND}$  finitely axiomatizable?

*Grazie*