Some Variants of Weak Pigeonhole Principle and WWKL

Wei Wang

A Joint Work with D. Bélanger, CT Chong, TL Wong and Y. Yang

Department of Philosophy
Sun Yat-sen University

Logic Colloquium, July 2018

Motivation

Theorem

- *(Avigad, Dean and Rute)* $\mathrm{RCA}_0 \vdash 2-\mathrm{WWKL}_0 \to 2-\mathrm{RAND}$, *where* $2-\mathrm{RAND}$ *asserts the existence of a 2-random real and* $2-\mathrm{WWKL}_0$ *is the statement that* $[T] \neq \emptyset$ *for every* Σ^0_2 *binary tree* T *s.t.*

$$\exists\delta > 0 \forall n(|T \cap 2^n| > \delta 2^n).$$

- *(ADR)* $\mathrm{RCA}_0 \vdash 2-\mathrm{WWKL}_0 \to B\Sigma^0_2$.
- *(Conidis and Slaman)* $2-\mathrm{WWKL}_0$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + B\Sigma^0_2$.
- *(CS)* $\mathrm{RCA}_0 \vdash 2-\mathrm{RAND} \to C\Sigma^0_2$, *where* $C\Sigma^0_2$ *states that every* Σ^0_2 *injection on all natural numbers has an unbounded range.*

Motivation

The Diagram

The arrows below are based on RCA_0:

$$\begin{array}{ccccc}
\mathsf{RCA}_0 & \longleftarrow & 2-\mathsf{RAND} & \longleftarrow & 2-\mathsf{WWKL}_0 \\
\downarrow & & \downarrow & & \downarrow \\
I\Sigma^0_1 & \longleftarrow & C\Sigma^0_2 & \longleftarrow & B\Sigma^0_2
\end{array}$$

An Obstacle

Theorem (Kučera)

If T is a binary tree with $\mu([T]) > 0$ and X is T-random then X computes some $Y \in [T]$.

As 2-random reals are $\emptyset'$-random, Kučera's Theorem implies that no models like $(\mathbb{N}, \mathcal{S})$ can separate $2 - \mathrm{WWKL}_0$ and $2 - \mathrm{RAND}$.

Theorem (Slaman, unpublished)

$\mathrm{RCA}_0 + 2 - \mathrm{RAND} \nvdash 2 - \mathrm{WWKL}_0$ *witnessed by non-standard models.*

An Interpolation

$2-\mathsf{WWKL}_0(1/2)$: $[T] \neq \emptyset$ for every Σ^0_2 binary tree T s.t.

$$\forall n(|T \cap 2^n| > 2^{n-1}).$$

If $\mathcal{M} = (\mathbb{N}, \mathcal{S}) \models \mathsf{RCA}_0$ then

$$\mathcal{M} \models 2-\mathsf{WWKL}_0 \Leftrightarrow \mathcal{M} \models 2-\mathsf{WWKL}_0(1/2).$$

Theorem (BCWWY)

$\mathsf{RCA}_0 \vdash 2-\mathsf{WWKL}_0 \rightarrow 2-\mathsf{WWKL}_0(1/2) \rightarrow 2-\mathsf{RAND}.$

Another Interpolation

Theorem (Dimitracopoulos and Paris)

Over $I\Sigma_n$ $(n > 0)$, $B\Sigma_{n+1}$ is equivalent to that there is no Σ_{n+1} injection from any $x + 1$ to x.

$\Sigma_n - \mathrm{WPHP}$: There is no Σ_{n+1} injection from any $2x$ to $x > 0$.

Obviously,

$$C\Sigma_{n+1} \leftarrow \Sigma_{n+1} - \mathrm{WPHP} \leftarrow B\Sigma_{n+1}.$$

The First Order Theory of $2 - \mathsf{WWKL}_0(1/2)$

Theorem (BCWWY)

1. $\mathsf{RCA}_0 \vdash 2 - \mathsf{WWKL}_0(1/2) \to \Sigma^0_2 - \mathsf{WPHP}$.
2. *Every countable* $\mathcal{M} \models \mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{WPHP}$ *can be expanded to a model of* $\mathsf{RCA}_0 + 2 - \mathsf{WWKL}_0(1/2)$*, so* $2 - \mathsf{WWKL}_0(1/2)$ *is* Π^1_1*-conservative over* $\mathsf{RCA}_0 + \Sigma^0_2 - \mathsf{WPHP}$ *and the first order theory of* $2 - \mathsf{WWKL}_0(1/2)$ *is axiomatized by* $I\Sigma_1 + \Sigma_2 - \mathsf{WPHP}$*.*

The second result is proved by forcing over a countable model $\mathcal{M}$ with conditions (σ, S), where σ is a finite (in $\mathcal{M}$) binary sequence and S is a Σ^0_2 binary tree with $\mu([S]) > 1/2$. The proof uses a restriction of Lebesgue's Density Theorem.

$\Sigma_{n+1} - \mathrm{WPHP}$ and $B\Sigma_{n+1}$

Theorem (BCWWY)

$I\Sigma_n + \Sigma_{n+1} - \mathrm{WPHP} \not\vdash B\Sigma_{n+1}$ $(n > 0)$.

The theorem depends on two technical results:

1. (BCWWY, Yokoyama) Let $M \models I\Sigma_n$ and I be a Δ_{n+1} proper cut in M. If $M \prec_{n+1} N$ and N has no element between I and $M - I$ then I is a Δ_{n+1} proper cut in N.
2. (BCWWY) Let $M \models I\Sigma_n$ and I be a Δ_{n+1} proper cut in M. If $a \in M$ and $F : 2a \to a$ is a $\Sigma_{n+1}(M)$-injection then there exists N s.t. $M \prec_{n+1} N \models I\Sigma_n$, N has no element between I and $M - I$, and N contains $c < 2a$ with $F(c)^N$ undefined.

The proof of the 2nd item above uses Paris' ultrapower construction: first carefully (!!!) constructs an ultrafilter U on M-finite subsets of $2a$ and then takes

$$N = M - \text{finite functions } f : 2a \to M/U.$$

$C\Sigma_{n+1}$ and $\Sigma_{n+1} - \mathrm{WPHP}$

Theorem (BCWWY)

$I\Sigma_n + C\Sigma_{n+1} \nvdash \Sigma_{n+1} - \mathrm{WPHP}$ ($n > 0$).

There are two proofs of the above theorem:

1. The 1st uses a so-called flexible formula constructed by Blanck and Enayat and obtains $N = \bigcup_k N_k \models C\Sigma_{n+1} + \neg\Sigma_{n+1} - \mathrm{WPHP}$, where $N_k \prec_{n+1,e} N_{k+1} \models \mathrm{PA}$. This can also be applied to get $I\Sigma_1 + \bigcup_{k\in\mathbb{N}} C\Sigma_k + \neg B\Sigma_2$, an unpublished theorem of Slaman by a similar proof.
2. The 2nd uses Paris' construction again to obtain a technical lemma: for countable $M \models I\Sigma_n$ and $1 < a \in M$ and a $\Sigma_{n+1}(M)$-injection $F : M \to b$, there is N s.t. $M \prec_{n+1,cf} N \models I\Sigma_n$, $[0,a]^M = [0,a]^N$ and F^N is partial; then it builds a chain of cofinal extensions beginning with a projection model of $I\Sigma_n$ (i.e., some $M \models I\Sigma_n$ having a Σ_{n+1} injection $P : M \to \mathbb{N}$).

A New Diagram

The arrows below are based on RCA_0:

$$
\begin{array}{ccccccc}
\mathrm{RCA}_0 & \longleftarrow & 2-\mathrm{RAND} & \longleftarrow & 2-\mathrm{WWKL}_0(1/2) & \longleftarrow & 2-\mathrm{WWKL}_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I\Sigma^0_1 & \longleftarrow & C\Sigma^0_2 & \longleftarrow & \Sigma^0_2-\mathrm{WPHP} & \longleftarrow & B\Sigma^0_2
\end{array}
$$

… and a Question

Is the first order theory of $2 - \mathrm{RAND}$ finitely axiomatizable?

Grazie