The effect of the HOD Hypothesis on the behavior of large cardinals from V in HOD

Yong Cheng

School of Philosophy
Wuhan University, Wuhan, China

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Question

1. *To what extent must a large cardinal in* V *exhibit its large cardinal properties in* HOD*?*
2. *To what extent does the existence of large cardinals in* V *imply the existence of large cardinals in* HOD*?*

Theorem

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Corollary

Suppose that κ has any of the following large cardinal properties

- Local: *weakly compact, indescribable, totally indescribable, Ramsey, strongly Ramsey, measurable, θ-tall, θ-strong, Woodin, θ-supercompact, superstrong, n-superstrong, ω-superstrong, λ-extendible, almost huge, huge, n-huge, rank-into-rank (I_0, I_1 and I_3);*
- Global: *unfoldable, strongly unfoldable, tall, strong, supercompact, superhuge, and many others.*

Then there is a forcing extension in which κ continues to have the property, but is not weakly compact in HOD.

Definition

(Woodin, [3]) Let λ be an uncountable regular cardinal. Then λ is ω-strongly measurable in HOD *iff there is $\kappa < \lambda$ such that* $(2^{\kappa})^{\mathrm{HOD}} < \lambda$ *and there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of $cof(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa \rangle \in \mathrm{HOD}$.*

The HOD Hypothesis

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Definition

(Woodin, [5]) The HOD *Hypothesis denotes the following statement: there is a proper class of regular cardinals that are not ω-strongly measurable in* HOD.

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1. *Whether under the* HOD *Hypothesis, behaviors of large cardinals from* V *become more regular in* HOD*?*
2. *Especially, whether and how, under the* HOD *Hypothesis, large cardinals in* V *can be transferred into* HOD*.*

In "The HOD Hypothesis and a supercompact cardinal, Yong Cheng, Mathematical Logic Quarterly", we answer this question for one supercompact cardinal and prove the following main result:

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In "The HOD Hypothesis and a supercompact cardinal, Yong Cheng, Mathematical Logic Quarterly", we answer this question for one supercompact cardinal and prove the following main result:

Theorem

If κ is supercompact and the HOD *Hypothesis holds, then there is a proper class of regular cardinals below κ which are measurable in* HOD*.*

The first motivation: The inner model program for one supercompact cardinal

Woodin shows in "Suitable extender models I" and "Suitable extender models II: Beyond ω-huge" in J. Math. Log that:

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2. Furthermore, if that construction of an L–like model is definable, then HOD must necessarily be close to V and the HOD Hypothesis must be true.

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2. Furthermore, if that construction of an L–like model is definable, then HOD must necessarily be close to V and the HOD Hypothesis must be true.

This motivates the HOD Hypothesis which is a good test question for the success of the inner model program for one supercompact cardinal.

The second motivation: The limits of the large cardinal hierarchy

- Most large cardinal hypotheses can be stated in terms of the existence of non-trivial elementary embeddings of the form $j : (\mathsf{V}, \in) \longrightarrow (\mathsf{M}, \in)$, where M is some transitive model.
- The closer the structure M is to V, the stronger is the large cardinal.

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Theorem

(Kunen, ZFC) *There does not exist a non-trivial elementary embedding* $j : (\mathrm{V}, \in) \longrightarrow (\mathrm{V}, \in)$.

It is open whether Kunen's theorem can be proved in ZF.

Definition

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Theorem

(Woodin, [3]) (ZF) Assume The HOD *Conjecture. Suppose δ is an extendible cardinal and $\lambda > \delta$.*[a] *Then there is no non-trivial elementary embedding $j : \mathrm{V}_{\lambda+2} \to \mathrm{V}_{\lambda+2}$.*

[a] δ is extendible if for every $\alpha > \delta$ there exist an ordinal β and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point δ.

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This theorem suggests that proving the HOD Conjecture would have a huge foundational significance: it would provide a route to show that there are no nontrivial elementary embeddings from V to V even if AC fails.

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Theorem

(Jensen, Dichotomy Theorem for L*) Exactly one of the following holds:*

1. L *is correct about singular cardinals and computes their successors correctly;*
2. *Every uncountable cardinal is inaccessible in* L*.*

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The following HOD Dichotomy Theorem can be seen as a generalization for HOD of Jensen's theorem for L.

The HOD Dichotomy Theorem

Theorem

(Woodin, HOD *Dichotomy Theorem)* *Assume that δ is an extendible cardinal. Then exactly one of the following holds.*

(A) *For every singular cardinal $\gamma > \delta$, γ is singular in* HOD *and* $\gamma^+ = (\gamma^+)^{\text{HOD}}$.

(B) *Every regular cardinal greater than δ is measurable in* HOD.

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(B) *Every regular cardinal greater than δ is measurable in* HOD.

- (A) says that HOD is close to V in the way that L is close to V when $0^\sharp$ does not exist
- (B) says that HOD is small compared to V also in very much the same way that L is small compared to V when $0^\sharp$ exists.
- The HOD Dichotomy Theorem motivates the HOD Hypothesis: The HOD Hypothesis rules out possibility (B) and therefore says that only (A) can be the case and therefore HOD is always close to V.

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1. *(Woodin) Suppose N is a proper class inner model of* V *and* δ *is a supercompact cardinal. Then* δ *is N-supercompact if for all* $\lambda > \delta$*, there exists an elementary embedding* $j : \mathrm{V} \to M$ *such that* $crit(j) = \delta, j(\delta) > \lambda, M^{\mathrm{V}_\lambda} \subseteq M$ *and* $j(N \cap \mathrm{V}_\delta) \cap \mathrm{V}_\lambda = N \cap \mathrm{V}_\lambda$.

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2. (Woodin) *Suppose N is a transitive class,* Ord $\subseteq N$ *and $N \models$ ZFC. N is a weak extender model for δ supercompact if for every $\gamma > \delta$ there exists a normal fine δ-complete measure $\mathcal{U}$ on $P_\delta(\gamma)$ such that $N \cap P_\delta(\gamma) \in \mathcal{U}$ and $\mathcal{U} \cap N \in N$.*

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2. *(Woodin) Suppose N is a transitive class,* $\mathrm{Ord} \subseteq N$ *and $N \models \mathrm{ZFC}$. N is a weak extender model for δ supercompact if for every $\gamma > \delta$ there exists a normal fine δ-complete measure $\mathcal{U}$ on $P_\delta(\gamma)$ such that $N \cap P_\delta(\gamma) \in \mathcal{U}$ and $\mathcal{U} \cap N \in N$.*
3. *For regular cardinals $\delta < \kappa$, we say (δ, κ) is a* HOD*-partition pair if there exists a partition $\langle S_\alpha \mid \alpha < \delta \rangle \in \mathrm{HOD}$ of $\{\alpha < \kappa \mid cf(\alpha) = \omega\}$ into pairwise disjoint stationary sets.*

The HOD Hypothesis under a HOD-supercompact cardinal

Theorem

(Woodin, [3]) Suppose δ is HOD-*supercompact. Then the following are equivalent:*

1. *The* HOD *Hypothesis.*
2. HOD *is a weak extender model for δ supercompact.*
3. *There exists a weak extender model N for δ supercompact such that $N \subseteq$* HOD*.*
4. *Every singular cardinal $\gamma > \delta$ is singular in* HOD *and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*
5. *There is a proper class of regular cardinals that are not measurable in* HOD*.*
6. *For any $\gamma > \delta$ there is regular cardinal $\lambda > \gamma$ such that (γ, λ) is a* HOD-*partition pair.*

The HOD Hypothesis under an extendible cardinal

Theorem

(Woodin, [3]) Suppose δ is extendible. Then the following are equivalent:

1. *The* HOD *Hypothesis.*
2. *There exists a regular cardinal $\kappa \geqslant \delta$ such that κ is not measurable in* HOD.
3. *There exists a regular cardinal $\kappa \geqslant \delta$ such that (δ, κ) is a* HOD*-partition pair.*
4. *For any cardinal κ, if κ is* HOD*-supercompact, then* HOD *is a weak extender model for κ-supercompact.*
5. *There exists a regular cardinal $\kappa \geqslant \delta$ such that κ is not ω-strongly measurable in* HOD.

Equivalences of supercompactness

Proposition

κ is supercompact if and only if for any $\lambda \geq \kappa$, there exists an elementary embedding $j : V \to M$ such that

1. $j(\gamma) = \gamma$ *for all* $\gamma < \kappa$;
2. $j(\kappa) > \lambda$;
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Theorem

(Magidor) δ *is supercompact if and only if for every* $\kappa > \delta$, *there exist* $\alpha < \delta$ *and an elementary embedding* $j : \mathrm{V}_\alpha \to \mathrm{V}_\kappa$ *with critical point* $\bar{\delta}$ *such that* $j(\bar{\delta}) = \delta$.

A new characterization of supercompactness

Theorem

κ is supercompact if and only if for all $\lambda > \kappa$, any $\alpha < \kappa$ and for all $N \subseteq V_\kappa$, there exist $\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa$, and elementary embeddings $\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \to V_{\lambda+1}$ such that

1. $\alpha < \kappa_1, crit(\pi_1) = \kappa_1$ *and* $crit(\pi_2) = \kappa_2$;
2. $\pi_2(\kappa_2) = \kappa$ *and* $\pi_1(\kappa_1) = \kappa_2$; *and*
3. $\pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2}$ *and* $\pi_2(N \cap V_{\lambda_2}) = N \cap V_\lambda$.

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Fix $\lambda > \kappa, \alpha < \kappa$ and $N \subseteq V_\kappa$.

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3. $\pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2}$ *and* $\pi_2(N \cap V_{\lambda_2}) = N \cap V_\lambda$.

Fix $\lambda > \kappa, \alpha < \kappa$ and $N \subseteq V_\kappa$.

- Take $j_0 : V \to M_0$ such that $crit(j_0) = \kappa$ and M_0 is closed under $V_{\lambda+1}$-sequences.

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Fix $\lambda > \kappa, \alpha < \kappa$ and $N \subseteq V_\kappa$.

- Take $j_0 : V \to M_0$ such that $crit(j_0) = \kappa$ and M_0 is closed under $V_{\lambda+1}$-sequences.
- Then $j_0(j_0) : M_0 \to M_1$ and M_1 is closed under $j_0(V_{\lambda+1})$-sequences in M_0. Let $j = j_0(j_0) \circ j_0$. Then $j : V \to M_1$.

Sketch of the proof

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- It suffices to show in M_1 that there exist $\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa)$, $\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \to V_{j(\lambda)+1}$ such that
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  (2) $\pi_2(\kappa_2) = j(\kappa)$ and $\pi_1(\kappa_1) = \kappa_2$; and
  (3) $\pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2}$ and $\pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda)}$.

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- Let $\kappa_1 = \kappa, \lambda_1 = \lambda, \kappa_2 = j_0(\kappa)$, $\lambda_2 = j_0(\lambda)$, $\pi_1 = j_0 \restriction V_{\lambda+1}$ and $\pi_2 = j_0(\pi_1) = j_0(j_0) \restriction j_0(V_{\lambda+1})$. Then $\pi_1 : V_{\lambda+1} \to V_{j_0(\lambda)+1}$ and $\pi_2 : V_{j_0(\lambda)+1} \to V_{j(\lambda)+1}$.

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- Since M_0 is closed under $\mathrm{V}_{\lambda+1}$-sequences in V, $\pi_1 \in M_1$. Since M_1 is closed under $j_0(\mathrm{V}_{\lambda+1})$-sequences in M_0, $\pi_2 \in M_1$.

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- Since M_0 is closed under $V_{\lambda+1}$-sequences in V, $\pi_1 \in M_1$. Since M_1 is closed under $j_0(V_{\lambda+1})$-sequences in M_0, $\pi_2 \in M_1$.
- It is easy to check that (1)-(3) holds.

Proof of the main theorem

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Suppose κ is supercompact and the HOD Hypothesis holds.

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- Find elementary embeddings $\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \to V_{\lambda+1}$ such that $crit(\pi_1) = \kappa_1, \alpha < \kappa_1 < \kappa$ and $\pi_3(\mathrm{HOD} \cap V_{\lambda_1}) \subseteq \mathrm{HOD}$ where $\pi_3 = \pi_2 \circ \pi_1$.

Proof of the main theorem

Theorem

Suppose κ is supercompact and the HOD *Hypothesis holds. Then for each $\alpha < \kappa$, there exists γ such that $\alpha < \gamma < \kappa$ and γ is measurable in* HOD.

Proof.

Suppose κ is supercompact and the HOD Hypothesis holds.

- Take $\alpha < \kappa$ and $\lambda > \kappa$ such that λ is a limit of regular cardinals which are not ω-strongly measurable in HOD and $\mathrm{HOD} \cap \mathrm{V}_\lambda = \mathrm{HOD}^{\mathrm{V}_\lambda}$.
- Find elementary embeddings $\pi_1 : \mathrm{V}_{\lambda_1+1} \to \mathrm{V}_{\lambda_2+1}$ and $\pi_2 : \mathrm{V}_{\lambda_2+1} \to \mathrm{V}_{\lambda+1}$ such that $\mathit{crit}(\pi_1) = \kappa_1, \alpha < \kappa_1 < \kappa$ and $\pi_3(\mathrm{HOD} \cap \mathrm{V}_{\lambda_1}) \subseteq \mathrm{HOD}$ where $\pi_3 = \pi_2 \circ \pi_1$.
- We show that κ_1 is measurable in HOD: it suffices to show that for any $\overline{\gamma} < \lambda_1, \pi_3 \restriction (\mathrm{HOD} \cap \mathrm{V}_{\overline{\gamma}}) \in \mathrm{HOD}$.

Sketch of the proof: Continued

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- Let $\pi_3(\overline{\gamma}) = \gamma$. Take $\delta > |\mathsf{V}_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and δ is not ω-strongly measurable in HOD.

Sketch of the proof: Continued

- Let $\pi_3(\overline{\gamma}) = \gamma$. Take $\delta > |\mathsf{V}_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and δ is not ω-strongly measurable in HOD.
- By the HOD Hypothesis, there exists $\langle S_\alpha \mid \alpha < |\mathsf{V}_{\gamma+\omega}|\rangle \in$ HOD which is a partition of S^δ_ω into stationary sets in δ.

Sketch of the proof: Continued

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- Since $\langle S_\alpha \mid \alpha < |V_{\gamma+\omega}|\rangle \in$ HOD, we can show that $\pi_3 \restriction |V_{\overline{\gamma}+\omega}| \in$ HOD.

Sketch of the proof: Continued

- Let $\pi_3(\overline{\gamma}) = \gamma$. Take $\delta > |V_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and δ is not ω-strongly measurable in HOD.
- By the HOD Hypothesis, there exists $\langle S_\alpha \mid \alpha < |V_{\gamma+\omega}|\rangle \in$ HOD which is a partition of S^δ_ω into stationary sets in δ.
- Since $\langle S_\alpha \mid \alpha < |V_{\gamma+\omega}|\rangle \in$ HOD, we can show that $\pi_3 \restriction |V_{\overline{\gamma}+\omega}| \in$ HOD.
- From $\pi_3(\mathrm{HOD} \cap V_{\lambda_1}) \subseteq$ HOD and $\pi_3 \restriction |V_{\overline{\gamma}+\omega}| \in$ HOD, by a standard argument, we can show that $\pi_3 \restriction (\mathrm{HOD} \cap V_{\overline{\gamma}}) \in$ HOD.

The following theorem shows the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis:

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Suppose κ is supercompact and the HOD *Hypothesis holds. Then there is a proper class of regular cardinals below κ which are measurable in* HOD.

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Corollary

Suppose κ is supercompact and the HOD *Hypothesis holds. Then there is a proper class of regular cardinals below κ which are measurable in* HOD*.*

Claim

(Woodin) If δ is HOD*-supercompact and the* HOD *Hypothesis holds, then any measurable cardinal $\kappa \geq \delta$ is measurable in* HOD*.*

Theorem

(Woodin, Global Universality Theorem, [3]) *Suppose the* HOD *Hypothesis holds and* δ *is* HOD*-supercompact. If* $j : \mathrm{HOD} \cap V_{\gamma+1} \to M \subseteq \mathrm{HOD} \cap V_{j(\gamma)+1}$ *is an elementary embedding with* $crit(j) \geq \delta$*. Then* $j \in \mathrm{HOD}$.

Global and Local Universality Theorem

Theorem

(Woodin, Global Universality Theorem, [3]) *Suppose the* HOD *Hypothesis holds and* δ *is* HOD*-supercompact. If* $j : \mathrm{HOD} \cap \mathrm{V}_{\gamma+1} \to M \subseteq \mathrm{HOD} \cap \mathrm{V}_{j(\gamma)+1}$ *is an elementary embedding with* $crit(j) \geq \delta$*. Then* $j \in \mathrm{HOD}$.

Corollary

(Woodin, Local Universality Theorem) *Suppose* κ *is supercompact and the* HOD *Hypothesis holds. Then for each* $\alpha < \kappa$*, there exists an elementary embedding* $j : \mathrm{V}_{\lambda+1} \to \mathrm{V}_{j(\lambda)+1}$ *such that*

1. $crit(j) = \overline{\kappa}$, $\alpha < \overline{\kappa} < \lambda < \kappa$ *and* $j(\lambda) < \kappa$;
2. $j \upharpoonright (\mathrm{HOD} \cap \mathrm{V}_{\lambda}) \in \mathrm{HOD}$ *and*
3. $j(\mathrm{HOD} \cap \mathrm{V}_{\lambda}) = \mathrm{HOD} \cap \mathrm{V}_{j(\lambda)}$.

Differences between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis

Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

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Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

1. Under the assumption of the HOD Hypothesis and HOD-supercompact cardinals, large cardinals in V are reflected to be large cardinals in HOD in a global way;

Differences between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis

Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

1. Under the assumption of the HOD Hypothesis and HOD-supercompact cardinals, large cardinals in V are reflected to be large cardinals in HOD in a global way;
2. However, under the assumption of the HOD Hypothesis and supercompact cardinals, large cardinals in V are reflected to be large cardinals in HOD in a local way.

Questions for future research

Under the assumption of only supercompactness, we do not know any equivalence of the HOD Hypothesis.

Question

1. *Whether we can establish the equivalence of the* HOD *Hypothesis only assuming supercompact cardinals. Especially, if κ is supercompact, whether the* HOD *Hypothesis is the equivalent to the statement: for each $\alpha < \kappa$, there exists γ such that $\alpha < \gamma < \kappa$ and γ is measurable in* HOD.
2. *What is the effect of the* HOD *Hypothesis on the behavior of large cardinals stronger than supercompact cardinals from* V *in* HOD. *Especially, assuming the* HOD *Hypothesis holds and δ is an extendible cardinal, whether the following holds: for any supercompact cardinal $\kappa \geq \delta$, κ is supercompact in* HOD*?*

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Thanks for your attention!