The effect of the HOD Hypothesis on the behavior of large cardinals from $V$ in $\text{HOD}$

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Examining how HOD and $V$ can be pushed apart via forcing and under which hypothesis HOD and $V$ are close to each other is a very interesting area of research.

Behaviors of large cardinals from $V$ can become disordered in HOD via forcing.

In "Large cardinals need not be large in HOD, Yong Cheng, Sy-David Friedman and Joel David Hamkins, Annals of Pure and Applied Logic", we examine the following questions:

1. To what extent must a large cardinal in $V$ exhibit its large cardinal properties in HOD?
2. To what extent does the existence of large cardinals in $V$ imply the existence of large cardinals in HOD?

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<thead>
<tr>
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<tbody>
<tr>
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1. *To what extent must a large cardinal in V exhibit its large cardinal properties in HOD?*

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Theorem

(Joint work with Sy-David Friedman and Joel David Hamkins) If $\kappa$ is a supercompact cardinal, then there is a forcing extension in which $\kappa$ remains supercompact, but is not weakly compact in HOD.
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Corollary

Suppose that $\kappa$ has any of the following large cardinal properties

**Local:** weakly compact, indescribable, totally indescribable, Ramsey, strongly Ramsey, measurable, $\theta$-tall, $\theta$-strong, Woodin, $\theta$-supercompact, superstrong, $n$-superstrong, $\omega$-superstrong, $\lambda$-extendible, almost huge, huge, $n$-huge, rank-into-rank ($l_0$, $l_1$, and $l_3$);

**Global:** unfoldable, strongly unfoldable, tall, strong, supercompact, superhuge, and many others.

Then there is a forcing extension in which $\kappa$ continues to have the property, but is not weakly compact in HOD.
(Woodin, [3]) Let $\lambda$ be an uncountable regular cardinal. Then $\lambda$ is $\omega$-strongly measurable in $\text{HOD}$ iff there is $\kappa < \lambda$ such that $(2^\kappa)^{\text{HOD}} < \lambda$ and there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of $\text{cof}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa \rangle \in \text{HOD}$.
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(Woodin, [5]) The HOD Hypothesis denotes the following statement: there is a proper class of regular cardinals that are not $\omega$-strongly measurable in HOD.
The main theorem

Question

Whether the HOD Hypothesis has some effect on the behavior of large cardinals from $V$ in HOD?

In "The HOD Hypothesis and a supercompact cardinal, Yong Cheng, Mathematical Logic Quarterly", we answer this question for one supercompact cardinal and prove the following main result:

**Theorem**

If $\kappa$ is supercompact and the HOD Hypothesis holds, then there is a proper class of regular cardinals below $\kappa$ which are measurable in HOD.
The main theorem

Question

Whether the HOD Hypothesis has some effect on the behavior of large cardinals from $V$ in HOD?

1. Whether under the HOD Hypothesis, behaviors of large cardinals from $V$ become more regular in HOD?

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Yong Cheng
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1. Whether under the HOD Hypothesis, behaviors of large cardinals from $V$ become more regular in HOD?

2. Especially, whether and how, under the HOD Hypothesis, large cardinals in $V$ can be transferred into HOD.

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In “The HOD Hypothesis and a supercompact cardinal, Yong Cheng, Mathematical Logic Quarterly”, we answer this question for one supercompact cardinal and prove the following main result:

Theorem

If κ is supercompact and the HOD Hypothesis holds, then there is a proper class of regular cardinals below κ which are measurable in HOD.
The first motivation: The inner model program for one supercompact cardinal

Woodin shows in “Suitable extender models I” and “Suitable extender models II: Beyond $\omega$-huge” in J. Math. Log that:

1. If the inner model program can be extended to prove that if there is a supercompact cardinal then there is a so–called $L$–like weak extender model for a supercompact cardinal, then that $L$–like model accommodates all large cardinal axioms that have ever been considered and is close to $V$ in a certain well–defined sense;

2. Furthermore, if that construction of an $L$–like model is definable, then $HOD$ must necessarily be close to $V$ and the $HOD$ Hypothesis must be true.

This motivates the $HOD$ Hypothesis which is a good test question for the success of the inner model program for one supercompact cardinal.
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The second motivation: The limits of the large cardinal hierarchy

- Most large cardinal hypotheses can be stated in terms of the existence of non-trivial elementary embeddings of the form $j : (V, \in) \rightarrow (M, \in)$, where $M$ is some transitive model.
- The closer the structure $M$ is to $V$, the stronger is the large cardinal.
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**Theorem**

*(Kunen, ZFC)* There does not exist a non-trivial elementary embedding $j : (V, \in) \rightarrow (V, \in)$.
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**Theorem**

(Kunen, ZFC) There does not exist a non-trivial elementary embedding \( j : (V, \in) \rightarrow (V, \in) \).

It is open whether Kunen’s theorem can be proved in ZF.
**Definition**

*The HOD Conjecture denotes the following statement: the theory ZFC + “there exists a supercompact cardinal” proves the HOD Hypothesis.*
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The HOD Conjecture denotes the following statement: the theory $\text{ZFC} + “\text{there exists a supercompact cardinal}”$ proves the HOD Hypothesis.

**Theorem**

(Woodin, [3]) (ZF) Assume The HOD Conjecture. Suppose $\delta$ is an extendible cardinal and $\lambda > \delta$.\(^a\) Then there is no non-trivial elementary embedding $j : V_{\lambda+2} \rightarrow V_{\lambda+2}$.

\(^a\delta$ is extendible if for every $\alpha > \delta$ there exist an ordinal $\beta$ and an elementary embedding $j : V_\alpha \rightarrow V_\beta$ with critical point $\delta$.\)
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$\delta$ is extendible if for every $\alpha > \delta$ there exist an ordinal $\beta$ and an elementary embedding $j : V_\alpha \rightarrow V_\beta$ with critical point $\delta$.

This theorem suggests that proving the HOD Conjecture would have a huge foundational significance: it would provide a route to show that there are no nontrivial elementary embeddings from $V$ to $V$ even if AC fails.
The third motivation: The HOD Dichotomy Theorem

The HOD Dichotomy says that either HOD is close to V or else HOD is far from V.

Theorem (Jensen, Dichotomy Theorem for $L$)

Exactly one of the following holds:

1. $L$ is correct about singular cardinals and computes their successors correctly;
2. Every uncountable cardinal is inaccessible in $L$.

The following HOD Dichotomy Theorem can be seen as a generalization for HOD of Jensen's theorem for $L$.
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The HOD Dichotomy Theorem

(Woodin, HOD Dichotomy Theorem) Assume that $\delta$ is an extendible cardinal. Then exactly one of the following holds.

(A) For every singular cardinal $\gamma > \delta$, $\gamma$ is singular in $\text{HOD}$ and $\gamma^+ = (\gamma^+)^{\text{HOD}}$.

(B) Every regular cardinal greater than $\delta$ is measurable in $\text{HOD}$.

The HOD Dichotomy Theorem motivates the HOD Hypothesis: The HOD Hypothesis rules out possibility (B) and therefore says that only (A) can be the case and therefore $\text{HOD}$ is always close to $V$. 

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The HOD Dichotomy Theorem

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(A) For every singular cardinal $\gamma > \delta$, $\gamma$ is singular in HOD and $\gamma^+ = (\gamma^+)^{\text{HOD}}$.

(B) Every regular cardinal greater than $\delta$ is measurable in HOD.

- (A) says that HOD is close to V in the way that L is close to V when $0^\#$ does not exist.
- (B) says that HOD is small compared to V also in very much the same way that L is small compared to V when $0^\#$ exists.

The HOD Dichotomy Theorem motivates the HOD Hypothesis: The HOD Hypothesis rules out possibility (B) and therefore says that only (A) can be the case and therefore HOD is always close to V.
Some key definitions

1. (Woodin) Suppose $N$ is a proper class inner model of $V$ and $\delta$ is a supercompact cardinal. Then $\delta$ is $N$-supercompact if for all $\lambda > \delta$, there exists an elementary embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \delta$, $j(\delta) > \lambda$, $M \subseteq V\lambda$ and $j(N \cap V\delta) \cap V\lambda = N \cap V\lambda$.

2. (Woodin) Suppose $N$ is a transitive class, $\text{Ord} \subseteq N$ and $N \models ZFC$. $N$ is a weak extender model for $\delta$ supercompact if for every $\gamma > \delta$ there exists a normal fine $\delta$-complete measure $U$ on $P\delta(\gamma)$ such that $N \cap P\delta(\gamma) \in U$ and $U \cap N \in N$.

3. For regular cardinals $\delta < \kappa$, we say $(\delta, \kappa)$ is a HOD-partition pair if there exists a partition $\langle S_\alpha | \alpha < \delta \rangle \in \text{HOD}$ of $\{ \alpha < \kappa | \text{cf}(\alpha) = \omega \}$ into pairwise disjoint stationary sets.
(Woodin) Suppose $N$ is a proper class inner model of $V$ and $\delta$ is a supercompact cardinal. Then $\delta$ is $N$-supercompact if for all $\lambda > \delta$, there exists an elementary embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \delta$, $j(\delta) > \lambda$, $M^{V^\lambda} \subseteq M$ and $j(N \cap V_\delta) \cap V_\lambda = N \cap V_\lambda$. 

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3. For regular cardinals $\delta < \kappa$, we say $(\delta, \kappa)$ is a HOD-partition pair if there exists a partition $\langle S_\alpha \mid \alpha < \delta \rangle \in \text{HOD}$ of $\{\alpha < \kappa \mid \text{cf}(\alpha) = \omega\}$ into pairwise disjoint stationary sets.
The HOD Hypothesis under a HOD-supercompact cardinal

Theorem

(Woodin, [3]) Suppose \( \delta \) is HOD-supercompact. Then the following are equivalent:

1. The HOD Hypothesis.
2. HOD is a weak extender model for \( \delta \) supercompact.
3. There exists a weak extender model \( N \) for \( \delta \) supercompact such that \( N \subseteq \text{HOD} \).
4. Every singular cardinal \( \gamma > \delta \) is singular in HOD and \( \gamma^+ = (\gamma^+)^{\text{HOD}} \).
5. There is a proper class of regular cardinals that are not measurable in HOD.
6. For any \( \gamma > \delta \) there is regular cardinal \( \lambda > \gamma \) such that \( (\gamma, \lambda) \) is a HOD-partition pair.
The HOD Hypothesis under an extendible cardinal

Theorem

(Woodin, [3]) Suppose $\delta$ is extendible. Then the following are equivalent:

1. The HOD Hypothesis.
2. There exists a regular cardinal $\kappa \geq \delta$ such that $\kappa$ is not measurable in HOD.
3. There exists a regular cardinal $\kappa \geq \delta$ such that $(\delta, \kappa)$ is a HOD-partition pair.
4. For any cardinal $\kappa$, if $\kappa$ is HOD-supercompact, then HOD is a weak extender model for $\kappa$-supercompact.
5. There exists a regular cardinal $\kappa \geq \delta$ such that $\kappa$ is not $\omega$-strongly measurable in HOD.

Yong Cheng
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Equivalences of supercompactness

Proposition

κ is supercompact if and only if for any λ ≥ κ, there exists an elementary embedding j : V → M such that

1. j(γ) = γ for all γ < κ;
2. j(κ) > λ;
3. M^λ ⊆ M; i.e., every sequence ⟨a_α : α < λ⟩ of elements of M is a member of M.
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Theorem

(Magidor) δ is supercompact if and only if for every κ > δ, there exist α < δ and an elementary embedding j : V_α → V_κ with critical point ¯δ such that j(¯δ) = δ.
Theorem

\( \kappa \) is supercompact if and only if for all \( \lambda > \kappa \), any \( \alpha < \kappa \) and for all \( N \subseteq V_\kappa \), there exist \( \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa \), and elementary embeddings \( \pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1} \) and \( \pi_2 : V_{\lambda_2+1} \to V_{\lambda+1} \) such that

1. \( \alpha < \kappa_1 \), \( \text{crit}(\pi_1) = \kappa_1 \) and \( \text{crit}(\pi_2) = \kappa_2 \);
2. \( \pi_2(\kappa_2) = \kappa \) and \( \pi_1(\kappa_1) = \kappa_2 \); and
3. \( \pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2} \) and \( \pi_2(N \cap V_{\lambda_2}) = N \cap V_{\lambda} \).
A new characterization of supercompactness

Theorem

\(\kappa\) is supercompact if and only if for all \(\lambda > \kappa\), any \(\alpha < \kappa\) and for all \(N \subseteq V_\kappa\), there exist \(\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa\), and elementary embeddings \(\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}\) and \(\pi_2 : V_{\lambda_2+1} \to V_{\lambda+1}\) such that

1. \(\alpha < \kappa_1\), \(\text{crit}(\pi_1) = \kappa_1\) and \(\text{crit}(\pi_2) = \kappa_2\);
2. \(\pi_2(\kappa_2) = \kappa\) and \(\pi_1(\kappa_1) = \kappa_2\); and
3. \(\pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2}\) and \(\pi_2(N \cap V_{\lambda_2}) = N \cap V_\lambda\).

Fix \(\lambda > \kappa\), \(\alpha < \kappa\) and \(N \subseteq V_\kappa\).
# A new characterization of supercompactness

## Theorem

\( \kappa \) is supercompact if and only if for all \( \lambda > \kappa \), any \( \alpha < \kappa \) and for all \( N \subseteq V_\kappa \), there exist \( \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa \), and elementary embeddings \( \pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1} \) and \( \pi_2 : V_{\lambda_2+1} \to V_{\lambda+1} \) such that

1. \( \alpha < \kappa_1 \), \( \text{crit}(\pi_1) = \kappa_1 \) and \( \text{crit}(\pi_2) = \kappa_2 \);
2. \( \pi_2(\kappa_2) = \kappa \) and \( \pi_1(\kappa_1) = \kappa_2 \); and
3. \( \pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2} \) and \( \pi_2(N \cap V_{\lambda_2}) = N \cap V_{\lambda} \).

Fix \( \lambda > \kappa \), \( \alpha < \kappa \) and \( N \subseteq V_\kappa \).

- Take \( j_0 : V \to M_0 \) such that \( \text{crit}(j_0) = \kappa \) and \( M_0 \) is closed under \( V_{\lambda+1} \)-sequences.
A new characterization of supercompactness

Theorem

\( \kappa \) is supercompact if and only if for all \( \lambda > \kappa \), any \( \alpha < \kappa \) and for all \( N \subseteq V_\kappa \), there exist \( \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa \), and elementary embeddings \( \pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1} \) and \( \pi_2 : V_{\lambda_2+1} \to V_{\kappa+1} \) such that

1. \( \alpha < \kappa_1, \text{crit}(\pi_1) = \kappa_1 \) and \( \text{crit}(\pi_2) = \kappa_2 \);
2. \( \pi_2(\kappa_2) = \kappa \) and \( \pi_1(\kappa_1) = \kappa_2 \); and
3. \( \pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2} \) and \( \pi_2(N \cap V_{\lambda_2}) = N \cap V_{\lambda} \).

Fix \( \lambda > \kappa, \alpha < \kappa \) and \( N \subseteq V_\kappa \).

- Take \( j_0 : V \to M_0 \) such that \( \text{crit}(j_0) = \kappa \) and \( M_0 \) is closed under \( V_{\lambda+1} \)-sequences.
- Then \( j_0(j_0) : M_0 \to M_1 \) and \( M_1 \) is closed under \( j_0(V_{\lambda+1}) \)-sequences in \( M_0 \). Let \( j = j_0(j_0) \circ j_0 \). Then \( j : V \to M_1 \).
Sketch of the proof

It suffices to show in $\mathcal{M}_1$ that there exist $\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa_1)$, $\pi_1: V_{\lambda_1 + 1} \to V_{\lambda_2 + 1}$ and $\pi_2: V_{\lambda_2 + 1} \to V_{j(\lambda_2) + 1}$ such that

1. $j(\alpha) = \alpha < \kappa_1$, crit($\pi_1$) = $\kappa_1$ and crit($\pi_2$) = $\kappa_2$;
2. $\pi_2(\kappa_2) = j(\kappa_1)$ and $\pi_1(\kappa_1) = \kappa_2$; and
3. $\pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2}$ and $\pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda_2)}$.

Let $\kappa_1 = \kappa, \lambda_1 = \lambda, \kappa_2 = j_0(\kappa), \lambda_2 = j_0(\lambda), \pi_1 = j_0|_{V_{\lambda_1 + 1}}$ and $\pi_2 = j_0(\pi_1) = j_0(j_0)$|$_{j_0(V_{\lambda_1 + 1})}$. Then $\pi_1: V_{\lambda_1 + 1} \to V_{j_0(\lambda_1) + 1}$ and $\pi_2: V_{j_0(\lambda_1) + 1} \to V_{j_0(\lambda_2) + 1}$.

Since $\mathcal{M}_0$ is closed under $V_{\lambda_1 + 1}$-sequences in $V$, $\pi_1 \in \mathcal{M}_1$. Since $\mathcal{M}_1$ is closed under $j_0(V_{\lambda_1 + 1})$-sequences in $\mathcal{M}_0$, $\pi_2 \in \mathcal{M}_1$.

It is easy to check that (1)-(3) holds.

Yong Cheng
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Sketch of the proof

It suffices to show in $M_1$ that there exist
$k_1 < \lambda_1 < k_2 < \lambda_2 < j(\kappa)$, $\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}$ and
$\pi_2 : V_{\lambda_2+1} \to V_{j(\lambda)+1}$ such that

1. $j(\alpha) = \alpha < k_1$, $crit(\pi_1) = k_1$ and $crit(\pi_2) = k_2$;
2. $\pi_2(k_2) = j(\kappa)$ and $\pi_1(k_1) = k_2$; and
3. $\pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2}$ and
   $\pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda)}$. 

Let $k_1 = k, \lambda_1 = \lambda, k_2 = j_0(k), \lambda_2 = j_0(\lambda)$, $\pi_1 = j_0|_{V_{\lambda_1+1}}$ and $\pi_2 = j_0(\pi_1) = j_0(j_0) \mid_{j_0(V_{\lambda_1+1})}$. Then
$\pi_1 : V_{\lambda_1+1} \to V_{j_0(\lambda)+1}$ and $\pi_2 : V_{j_0(\lambda)+1} \to V_{j(\lambda)+1}$.

Since $M_0$ is closed under $V_{\lambda_1+1}$-sequences in $V$, $\pi_1 \in M_1$.
Since $M_1$ is closed under $j_0(V_{\lambda_1+1})$-sequences in $M_0$, $\pi_2 \in M_1$.
It is easy to check that (1)-(3) holds.
Sketch of the proof

- It suffices to show in $M_1$ that there exist $\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa)$, $\pi_1 : V_{\lambda_1+1} \to V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \to V_{j(\lambda)+1}$ such that
  1. $j(\alpha) = \alpha < \kappa_1$, $\text{crit}(\pi_1) = \kappa_1$ and $\text{crit}(\pi_2) = \kappa_2$;
  2. $\pi_2(\kappa_2) = j(\kappa)$ and $\pi_1(\kappa_1) = \kappa_2$; and
  3. $\pi_1(j(\mathcal{N}) \cap V_{\lambda_1}) = j(\mathcal{N}) \cap V_{\lambda_2}$ and $\pi_2(j(\mathcal{N}) \cap V_{\lambda_2}) = j(\mathcal{N}) \cap V_{j(\lambda)}$.

- Let $\kappa_1 = \kappa$, $\lambda_1 = \lambda$, $\kappa_2 = j_0(\kappa)$, $\lambda_2 = j_0(\lambda)$, $\pi_1 = j_0 \upharpoonright V_{\lambda+1}$ and $\pi_2 = j_0(\pi_1) = j_0(j_0) \upharpoonright j_0(V_{\lambda+1})$. Then $\pi_1 : V_{\lambda+1} \to V_{j_0(\lambda)+1}$ and $\pi_2 : V_{j_0(\lambda)+1} \to V_{j(\lambda)+1}$. 

Since $M_0$ is closed under $V_{\lambda+1}$-sequences in $V$, $\pi_1 \in M_1$.

Since $M_1$ is closed under $j_0(V_{\lambda+1})$-sequences in $M_0$, $\pi_2 \in M_1$.

It is easy to check that (1)-(3) holds.
Sketch of the proof

- It suffices to show in $M_1$ that there exist
  $\kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa)$, $\pi_1 : V_{\lambda_1 + 1} \rightarrow V_{\lambda_2 + 1}$ and
  $\pi_2 : V_{\lambda_2 + 1} \rightarrow V_{j(\lambda) + 1}$ such that
  (1) $j(\alpha) = \alpha < \kappa_1$, $\text{crit}(\pi_1) = \kappa_1$ and $\text{crit}(\pi_2) = \kappa_2$;
  (2) $\pi_2(\kappa_2) = j(\kappa)$ and $\pi_1(\kappa_1) = \kappa_2$; and
  (3) $\pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2}$ and
   $\pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda)}$.

- Let $\kappa_1 = \kappa$, $\lambda_1 = \lambda$, $\kappa_2 = j_0(\kappa)$, $\lambda_2 = j_0(\lambda)$, $\pi_1 = j_0 \upharpoonright V_{\lambda + 1}$
  and $\pi_2 = j_0(\pi_1) = j_0(j_0) \upharpoonright j_0(V_{\lambda + 1})$. Then
  $\pi_1 : V_{\lambda + 1} \rightarrow V_{j_0(\lambda) + 1}$ and $\pi_2 : V_{j_0(\lambda) + 1} \rightarrow V_{j(\lambda) + 1}$.

- Since $M_0$ is closed under $V_{\lambda + 1}$-sequences in $V$, $\pi_1 \in M_1$.
  Since $M_1$ is closed under $j_0(V_{\lambda + 1})$-sequences in $M_0$, $\pi_2 \in M_1$. 
Sketch of the proof

- It suffices to show in $M_1$ that there exist
  \[ \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa), \quad \pi_1 : V_{\lambda_1 + 1} \rightarrow V_{\lambda_2 + 1} \text{ and} \]
  \[ \pi_2 : V_{\lambda_2 + 1} \rightarrow V_{j(\lambda) + 1} \]
  such that
  \begin{enumerate}
    \item $j(\alpha) = \alpha < \kappa_1$, $\text{crit}(\pi_1) = \kappa_1$ and $\text{crit}(\pi_2) = \kappa_2$;
    \item $\pi_2(\kappa_2) = j(\kappa)$ and $\pi_1(\kappa_1) = \kappa_2$; and
    \item $\pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2}$ and
      $\pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda)}$.
  \end{enumerate}

- Let $\kappa_1 = \kappa$, $\lambda_1 = \lambda$, $\kappa_2 = j_0(\kappa)$, $\lambda_2 = j_0(\lambda)$, $\pi_1 = j_0 \upharpoonright V_{\lambda + 1}$ and $\pi_2 = j_0(\pi_1) = j_0(j_0) \upharpoonright j_0(V_{\lambda + 1})$. Then
  $\pi_1 : V_{\lambda + 1} \rightarrow V_{j_0(\lambda) + 1}$ and $\pi_2 : V_{j_0(\lambda) + 1} \rightarrow V_{j(\lambda) + 1}$.

- Since $M_0$ is closed under $V_{\lambda + 1}$-sequences in $V$, $\pi_1 \in M_1$. Since $M_1$ is closed under $j_0(V_{\lambda + 1})$-sequences in $M_0$, $\pi_2 \in M_1$.

- It is easy to check that (1)-(3) holds.
Proof of the main theorem

Theorem

Suppose \( \kappa \) is supercompact and the HOD Hypothesis holds. Then for each \( \alpha < \kappa \), there exists \( \gamma \) such that \( \alpha < \gamma < \kappa \) and \( \gamma \) is measurable in HOD.
Theorem

Suppose \( \kappa \) is supercompact and the HOD Hypothesis holds. Then for each \( \alpha < \kappa \), there exists \( \gamma \) such that \( \alpha < \gamma < \kappa \) and \( \gamma \) is measurable in HOD.

Proof.

Suppose \( \kappa \) is supercompact and the HOD Hypothesis holds.
Proof of the main theorem

**Theorem**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then for each $\alpha < \kappa$, there exists $\gamma$ such that $\alpha < \gamma < \kappa$ and $\gamma$ is measurable in HOD.

**Proof.**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds.

- Take $\alpha < \kappa$ and $\lambda > \kappa$ such that $\lambda$ is a limit of regular cardinals which are not $\omega$-strongly measurable in HOD and $\text{HOD} \cap V_\lambda = \text{HOD}^V_\lambda$. 

Yong Cheng

The effect of the HOD Hypothesis on the behavior of large cardinals
Proof of the main theorem

**Theorem**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then for each $\alpha < \kappa$, there exists $\gamma$ such that $\alpha < \gamma < \kappa$ and $\gamma$ is measurable in HOD.

**Proof.**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds.

- Take $\alpha < \kappa$ and $\lambda > \kappa$ such that $\lambda$ is a limit of regular cardinals which are not $\omega$-strongly measurable in HOD and $\text{HOD} \cap V_\lambda = \text{HOD}^{V_\lambda}$.
- Find elementary embeddings $\pi_1 : V_{\lambda_1+1} \rightarrow V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \rightarrow V_{\lambda+1}$ such that $\text{crit}(\pi_1) = \kappa_1$, $\alpha < \kappa_1 < \kappa$ and $\pi_3(\text{HOD} \cap V_{\lambda_1}) \subseteq \text{HOD}$ where $\pi_3 = \pi_2 \circ \pi_1$. 

Yong Cheng

The effect of the HOD Hypothesis on the behavior of large cardinals
Proof of the main theorem

**Theorem**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then for each $\alpha < \kappa$, there exists $\gamma$ such that $\alpha < \gamma < \kappa$ and $\gamma$ is measurable in HOD.

**Proof.**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds.

- Take $\alpha < \kappa$ and $\lambda > \kappa$ such that $\lambda$ is a limit of regular cardinals which are not $\omega$-strongly measurable in HOD and $\text{HOD} \cap V_\lambda = \text{HOD}^V_\lambda$.

- Find elementary embeddings $\pi_1 : V_{\lambda_1+1} \rightarrow V_{\lambda_2+1}$ and $\pi_2 : V_{\lambda_2+1} \rightarrow V_{\lambda+1}$ such that $\text{crit}(\pi_1) = \kappa_1$, $\alpha < \kappa_1 < \kappa$ and $\pi_3(\text{HOD} \cap V_{\lambda_1}) \subseteq \text{HOD}$ where $\pi_3 = \pi_2 \circ \pi_1$.

- We show that $\kappa_1$ is measurable in HOD: it suffices to show that for any $\overline{\gamma} < \lambda_1$, $\pi_3 \upharpoonright (\text{HOD} \cap V_{\overline{\gamma}}) \in \text{HOD}$. 

The effect of the HOD Hypothesis on the behavior of large cardinals
Sketch of the proof: Continued

Let \( \pi_3(\gamma) = \gamma \). Take \( \delta > |V_\gamma + \omega + 1| \) such that \( \delta < \lambda \) and \( \delta \) is not \( \omega \)-strongly measurable in HOD.

By the HOD Hypothesis, there exists \( \langle S_\alpha | \alpha < |V_\gamma + \omega | \rangle \in \text{HOD} \) which is a partition of \( S_\delta \omega \) into stationary sets in \( \delta \).

Since \( \langle S_\alpha | \alpha < |V_\gamma + \omega | \rangle \in \text{HOD} \), we can show that \( \pi_3|_{|V_\gamma + \omega |} \in \text{HOD} \).

From \( \pi_3(\text{HOD} \cap V_{\lambda 1}) \subseteq \text{HOD} \) and \( \pi_3|_{|V_\gamma + \omega |} \in \text{HOD} \), by a standard argument, we can show that \( \pi_3|((\text{HOD} \cap V_\gamma) \cup |V_\gamma + \omega |) \in \text{HOD} \).
Let $\pi_3(\bar{\gamma}) = \gamma$. Take $\delta > |V_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and $\delta$ is not $\omega$-strongly measurable in HOD.
Let $\pi_3(\gamma) = \gamma$. Take $\delta > |V_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and $\delta$ is not $\omega$-strongly measurable in HOD.

By the HOD Hypothesis, there exists
\[ \langle S_\alpha \mid \alpha < |V_{\gamma+\omega}| \rangle \in \text{HOD} \]
which is a partition of $S^\delta_\omega$ into stationary sets in $\delta$. 
Let $\pi_3(\gamma) = \gamma$. Take $\delta > |V_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and $\delta$ is not $\omega$-strongly measurable in HOD.

By the HOD Hypothesis, there exists $\langle S_\alpha | \alpha < |V_{\gamma+\omega}| \rangle \in HOD$ which is a partition of $S_\omega^\delta$ into stationary sets in $\delta$.

Since $\langle S_\alpha | \alpha < |V_{\gamma+\omega}| \rangle \in HOD$, we can show that $\pi_3 \upharpoonright |V_{\gamma+\omega}| \in HOD$. 
Let $\pi_3(\bar{\gamma}) = \gamma$. Take $\delta > |V_{\gamma+\omega+1}|$ such that $\delta < \lambda$ and $\delta$ is not $\omega$-strongly measurable in HOD.

By the HOD Hypothesis, there exists $\langle S_{\alpha} \mid \alpha < |V_{\gamma+\omega}| \rangle \in \text{HOD}$ which is a partition of $S^\delta_\omega$ into stationary sets in $\delta$.

Since $\langle S_{\alpha} \mid \alpha < |V_{\gamma+\omega}| \rangle \in \text{HOD}$, we can show that $\pi_3 \upharpoonright |V_{\gamma+\omega}| \in \text{HOD}$.

From $\pi_3(\text{HOD} \cap V_{\lambda_1}) \subseteq \text{HOD}$ and $\pi_3 \upharpoonright |V_{\gamma+\omega}| \in \text{HOD}$, by a standard argument, we can show that $\pi_3 \upharpoonright (\text{HOD} \cap V_{\bar{\gamma}}) \in \text{HOD}$. 
The following theorem shows the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis:

Corollary

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then there is a proper class of regular cardinals below $\kappa$ which are measurable in HOD.

Claim (Woodin) If $\delta$ is HOD-supercompact and the HOD Hypothesis holds, then any measurable cardinal $\kappa \geq \delta$ is measurable in HOD.
The following theorem shows the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis:

**Corollary**

*Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then there is a proper class of regular cardinals below $\kappa$ which are measurable in HOD.*
The following theorem shows the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis:

**Corollary**

Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then there is a proper class of regular cardinals below $\kappa$ which are measurable in HOD.

**Claim**

(Woodin) If $\delta$ is HOD-supercompact and the HOD Hypothesis holds, then any measurable cardinal $\kappa \geq \delta$ is measurable in HOD.
Global and Local Universality Theorem

**Theorem**

(Woodin, Global Universality Theorem, [3]) Suppose the HOD Hypothesis holds and $\delta$ is HOD-supercompact. If $j : \text{HOD} \cap V_{\gamma+1} \rightarrow M \subseteq \text{HOD} \cap V_{j(\gamma)+1}$ is an elementary embedding with $\text{crit}(j) \geq \delta$. Then $j \in \text{HOD}$.
**Theorem**

(Woodin, Global Universality Theorem, [3]) Suppose the HOD Hypothesis holds and $\delta$ is HOD-supercompact. If $j : HOD \cap V_{\gamma+1} \to M \subseteq HOD \cap V_{j(\gamma)+1}$ is an elementary embedding with $\text{crit}(j) \geq \delta$. Then $j \in HOD$.

**Corollary**

(Woodin, Local Universality Theorem) Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then for each $\alpha < \kappa$, there exists an elementary embedding $j : V_{\lambda+1} \to V_{j(\lambda)+1}$ such that

1. $\text{crit}(j) = \kappa$, $\alpha < \kappa < \lambda < \kappa$ and $j(\lambda) < \kappa$;
2. $j \upharpoonright (HOD \cap V_{\lambda}) \in HOD$ and
3. $j(HOD \cap V_{\lambda}) = HOD \cap V_{j(\lambda)}$. 

Yong Cheng

The effect of the HOD Hypothesis on the behavior of large cardinals
Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

1. Under the assumption of the HOD Hypothesis and HOD-supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in $\text{HOD}$ in a global way;
2. However, under the assumption of the HOD Hypothesis and supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in $\text{HOD}$ in a local way.
Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

1. Under the assumption of the HOD Hypothesis and HOD-supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in HOD in a global way;
Differences between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis

Even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy, under the HOD Hypothesis, there is huge difference between HOD-supercompact cardinals and supercompact cardinals:

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2. However, under the assumption of the HOD Hypothesis and supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in HOD in a local way.

Yong Cheng

The effect of the HOD Hypothesis on the behavior of large cardinals
Under the assumption of only supercompactness, we do not know any equivalence of the HOD Hypothesis.

**Question**

1. Whether we can establish the equivalence of the HOD Hypothesis only assuming supercompact cardinals. Especially, if $\kappa$ is supercompact, whether the HOD Hypothesis is the equivalent to the statement: for each $\alpha < \kappa$, there exists $\gamma$ such that $\alpha < \gamma < \kappa$ and $\gamma$ is measurable in HOD.

2. What is the effect of the HOD Hypothesis on the behavior of large cardinals stronger than supercompact cardinals from $V$ in HOD. Especially, assuming the HOD Hypothesis holds and $\delta$ is an extendible cardinal, whether the following holds: for any supercompact cardinal $\kappa \geq \delta$, $\kappa$ is supercompact in HOD?


Thanks for your attention!