WHAT CANNOT BE SOLVED BY THE ELLIPSOID METHOD?

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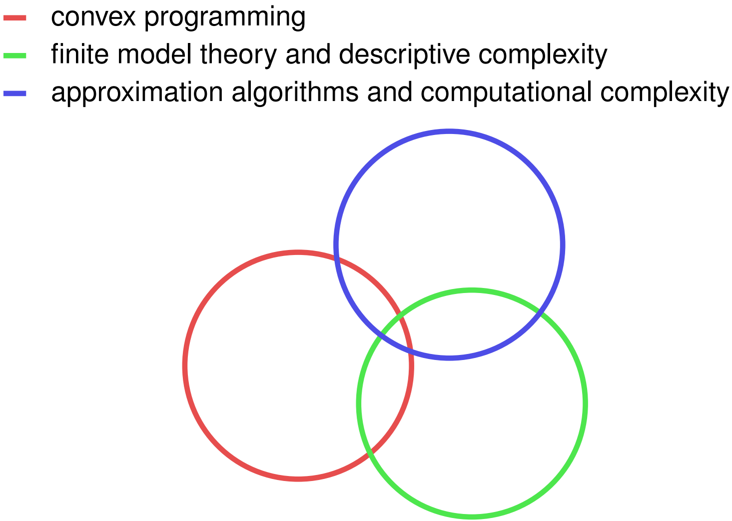

convex programming
finite model theory and descriptive complexity
approximation algorithms and computational complexity

Part I

ELLIPSOID METHOD

The Ellipsoid Method

- Invented for non-linear convex optimization over $\mathbb{R}^n$ in 1970's.
- Adapted to linear programming (LP) by Khachiyan in 1979.

$$\begin{array}{ll} \text{Feasibility:} & \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0 \\ \text{Optimization:} & \max\ \ \mathbf{c}^{\mathrm{T}}\mathbf{x}\ \ \text{s.t.}\ \ \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0. \end{array}$$

- First poly-time algorithm for LP: solved a big theoretical problem.
- Time poly in $\mathrm{size}(\mathbf{A})$, $\mathrm{size}(\mathbf{b})$, $\mathrm{size}(\mathbf{c})$ in bit-model of computation.

Problem Statement

Given: a convex $P \subseteq \mathbb{R}^n$ and an accuracy parameter $\epsilon > 0$.

Goal: find some point **x** in P.

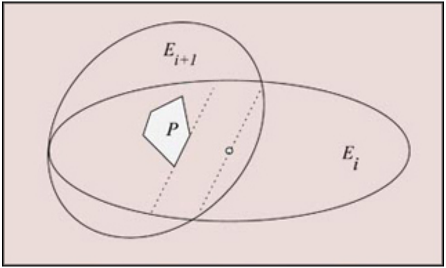
Assumptions:

- promise that $P \subseteq S(\mathbf{0}, R)$ for some known $R > 0$,
- promise that $S(\mathbf{x}_0, r) \subseteq P$ for some unknown $\mathbf{x}_0$ and $r > 0$,
- promise that a separation oracle for P is available.

Algorithm and Convergence

Start: $P \subseteq E_0 := S(0, R)$

Steps: $i = 0, 1, 2, \ldots$

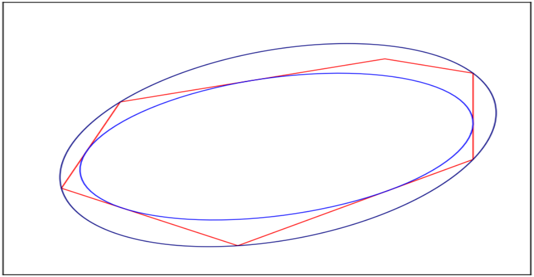


Progress: $\mathrm{vol}(E_{i+1}) \leq \left(1 - \frac{1}{\mathrm{poly}(n)}\right) \mathrm{vol}(E_i)$

Terminate: either $\mathrm{center}(E_i) \in P$ or $\mathrm{vol}(E_i) \leq \mathrm{vol}(S(x_0, r))$

Geometric Basis for Progress Measure

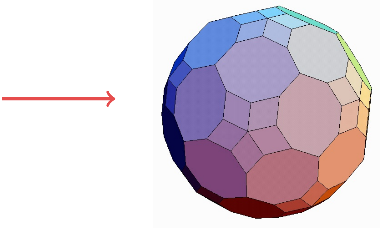
The Löwner-John ellipsoid



Theorem: For every convex $P \subseteq \mathbb{R}^n$, there is a unique ellipsoid E of minimial volume containing K. Moreover, K contains E shrinked by a factor of n.

Linear and Semidefinite Programming (LP and SDP)

$$\begin{array}{ll} \text{maximize} & \langle \mathbf{c}, \mathbf{x} \rangle \\ \text{subject to} & \langle \mathbf{a}_j, \mathbf{x} \rangle = b_j, \ j \in [m] \\ & \mathbf{x} \geq 0 \end{array}$$



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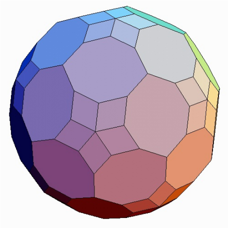

maximize $\langle \mathbf{C}, \mathbf{X} \rangle$
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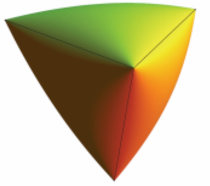
$\mathbf{X}$ is positive semi-definite (PSD)



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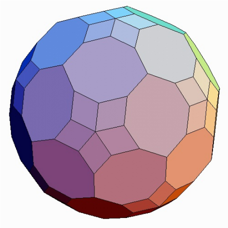

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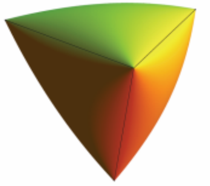

Part II

LP AND SDP FOR COMBINATORICS

Vertex cover

Problem:

Given an undirected graph $G = (V, E)$,
find the smallest number of vertices
that touches every edge.

Notation:

$$\mathrm{vc}(G).$$

Observe:

$A \subseteq V$ is a vertex cover of G
iff
$V \setminus A$ is an independent set of G

Linear programming relaxation

LP relaxation:

minimize $\sum_{u \in V} x_u$

subject to

$x_u + x_v \geq 1$ for every $(u, v) \in E$,

$x_u \geq 0$ for every $u \in V$.

Notation:

$$\mathrm{fvc}(G).$$

Approximation

Approximation:

$$\mathrm{fvc}(G) \leq \mathrm{vc}(G) \leq 2 \cdot \mathrm{fvc}(G)$$

Integrality gap:

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{fvc}(G)}$$

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Gap examples:

1. $\mathrm{vc}(K_n) = n - 1$,
2. $\mathrm{fvc}(K_n) = \frac{1}{2}n$.

LP tightenings

Add triangle inequalities:

minimize $\sum_{u \in V} x_u$

subject to

$x_u + x_v \geq 1$ for every $(u, v) \in E$,

$x_u \geq 0$ for every $u \in V$,

$x_u + x_v + x_w \geq 2$ for every triangle $\{u, v, w\}$ in G.

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Integrality gap:

Remains 2.

Gap examples:

Triangle-free graphs with small independence number.

LP and SDP Hierarchies

Hierarchy:

Systematic ways of
generating all linear inequalities
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$$P = \{x \in \mathbb{R}^n : Ax \geq b\},$$
$$P^{\mathbb{Z}} = \text{convexhull}\{x \in \{0,1\}^n : Ax \geq b\}.$$

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Produce explicit nested polytopes:

$$P = P^1 \supseteq P^2 \supseteq \cdots \supseteq P^{n-1} \supseteq P^n = P^{\mathbb{Z}}$$

P^k: SDP Hierarchy (Lasserre/SOS Hierarchy)

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Then:

$$P^k = \{x \in \mathbb{R}^n : L(x) \geq 0 \text{ for each produced } L \geq 0\}$$

P^k: LP Hierarchy (Sherali-Adams Hierarchy)

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where

$$Q_i = \sum_{\ell \in J} c_\ell \prod_{i \in A_\ell} x_i \prod_{i \in B_\ell} (1 - x_i) \quad \text{with} \quad c_\ell \geq 0$$

and

$$\deg(Q_0), \deg(L_j Q_j), \deg((x_i^2 - x_i) Q_i) \leq k.$$

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Example: triangles in P^3

For each triangle $\{u, v, w\}$ in G:

$$
\begin{aligned}
&Q_0 + \\
&(x_u + x_v - 1)Q_1 + \\
&(x_u + x_w - 1)Q_2 + \\
&(x_v + x_w - 1)Q_3 + \\
&(x_u^2 - x_u)Q_4 + \\
&(x_v^2 - x_v)Q_5 + \\
&(x_w^2 - x_w)Q_6 \\
&= ? \\
&(x_u + x_v + x_w - 2).
\end{aligned}
$$

$$Q_i = a_i + b_i x_u + c_i x_v + d_i x_w + e_i x_u x_v + f_i x_u x_w + g_i x_v x_w + h_i x_u x_v x_w$$

Solving P^k

Lift-and-project:

- Step 1: lift from $\mathbb{R}^n$ up to $\mathbb{R}^{(n+1)^k}$ and linearize the problem
- Step 2: project from $\mathbb{R}^{(n+1)^k}$ down to $\mathbb{R}^n$

Proposition:

Optimization of linear functions over P^k
can be solved in time[†] $m^{O(1)} n^{O(k)}$.

Proof:

1. for LP-P^k: by linear programming
2. for SDP-P^k: by semidefinite programming

An Important Open Problem

Define

$\mathrm{sa}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of LP-P^k
$\mathrm{sdp}^k\mathrm{fvc}(G)$: optimum fractional vertex cover of SDP-P^k

Open problem:

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{sdp}^4\mathrm{fvc}(G)} \stackrel{?}{<} 2$$

What's Known

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Known (conditional hardness):

- 1.0001-approximating $\mathrm{vc}(G)$ is NP-hard by PCP Theorem
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- $\sup_G \mathrm{vc}(G)/\mathrm{sa}^k\mathrm{fvc}(G) = 2$ for any $k = n^{o(1)}$
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Gap examples:

Frankl-Rödl Graphs: $\mathrm{FR}^n_\gamma = (\mathbb{F}_2^n, \{\{x, y\} : x + y \in A^n_\gamma\})$.

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Frankl-Rödl Graphs: $\mathrm{FR}_\gamma^n = (\mathbb{F}_2^n, \{\{x, y\} : x + y \in A_\gamma^n\})$.

[Dinur, Safra, Khot, Regev, Kleinberg, Charikar, Hatami, Magen, Georgiou, Lovasz, Arora, Alekhnovich, Pitassi; 2000's]

Part III

COUNTING LOGIC

Bounded-Variable Logics

First-order logic of graphs:

$E(x, y)$	:	x and y are joined by an edge
$x = y$	:	x and y denote the same vertex
$\neg\phi$	:	negation of ϕ holds
$\phi \wedge \psi$	:	both ϕ and ψ hold
$\exists x(\phi)$	:	there exists a vertex x that satisfies ϕ

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First-order logic with k variables (or width k) :

L^k: collection of formulas for which
all subformulas have at most k free variables.

Example

Paths:

$$
\begin{aligned}
P_1(x,y) &:= E(x,y) \\
P_2(x,y) &:= \exists z_1 (E(x,z_1) \wedge P_1(z_1,y)) \\
P_3(x,y) &:= \exists z_2 (E(x,z_2) \wedge P_2(z_2,y)) \\
&\vdots \\
P_{i+1}(x,y) &:= \exists z_i (E(x,z_i) \wedge P_i(z_i,y)) \\
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Bipartiteness of n-vertex graphs:

$$\forall x(\neg P_3(x,x) \wedge \neg P_5(x,x) \wedge \cdots \wedge \neg P_{2\lceil n/2 \rceil -1}(x,x)).$$

Counting quantifiers

Counting witnesses:

$\exists^{\geq i} x(\phi(x))$: there are at least i vertices x that satisfy $\phi(x)$.

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with all subformulas with at most k free variables.

Indistinguishability / Elementary equivalence

C^k**-equivalence**:

$G \equiv_k^C H$: G and H satisfy the same sentences of C^k.

Combinatorial characterization of C^2-equivalence

Color-refinement:

1. color each vertex black,
2. color each vertex by number of neighbors in each color-class,
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Theorem [Immerman and Lander]

$$G \equiv^C_2 H \text{ if and only if } G \equiv^R H$$

LP characterization of color-refinement

Isomorphisms:

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Theorem [Tinhofer]

$G \equiv^R H$ if and only if $G \equiv^F H$.

Higher levels of LP Hierarchy

LP-levels of fractional isomorphism:

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Moreover:

1. This interleaving is strict for $k > 2$ [Grohe-Otto 2015]
2. A combined LP characterizes $\equiv_k^C$ exactly [Grohe-Otto 2015]
3. Alternative (and independent) formulation by [Malkin 2014]

Higher Levels of SDP Hierarchy

LP and SDP-levels of fractional isomorphism:

1. $G \equiv_k^{\mathrm{LP}} H$: the degree-k LP level of $\mathrm{iso}(G, H)$ is feasible.
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Theorem [AA and Ochremiak 2018]

$$G \equiv_{ck}^{\mathrm{LP}} H \Longrightarrow G \equiv_k^{\mathrm{SDP}} H \Longrightarrow G \equiv_k^{\mathrm{LP}} H.$$

Proof-existence Problem for degree-k SDP

Theorem [AA and Ochremiak 2018]

For every $k \geq 1$ there is a set Φ_k of C$^{O(k)}$-formulas s.t. for every system of polyn. eqns. S including $X_i^2 - X_i$'s:

$$S \models \Phi_k \iff S \text{ has a degree-}k \text{ SDP refutation.}$$

How?

By reduction to feasibility of SDPs:

$$\langle A_j, X\rangle = b_j,\ j \in [m],$$
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How? By formalizing the ellipsoid method: [ADH15].

Part IV

BACK TO VERTEX COVER

Back to integrality gaps for vertex cover

Goal:

For large k and every $\epsilon > 0$ find graphs G and H such that

1. $G \equiv^{C}_{c \cdot 2k} H$
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Proof:

$$\begin{aligned}
\mathrm{vc}(G) &\geq (2-\epsilon)\mathrm{vc}(H) && \text{by 2.} \\
&\geq (2-\epsilon)\mathrm{sdp}^k\mathrm{fvc}(H) && \text{obvious} \\
&\geq (2-\epsilon)\mathrm{sdp}^k\mathrm{fvc}(G) && \text{by 1. and definability}
\end{aligned}$$

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A weak (easy) case: $k = 1$ with gap = 2

Choose:

G = any d-regular expander graph (i.e., $\lambda_2(G) \ll \lambda_1(G)$),
H = any d-regular bipartite graph.

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Then:

$\mathrm{vc}(G) = (1-\epsilon)n$	by expansion
$\mathrm{vc}(H) = n/2$	by bipartition
$G \equiv^R H$	by regularity
$G \equiv^C_2 H$	by Tinhofer's Theorem

A weak (easy) case: $k = 1$ with gap = 2

Choose:

G = any d-regular expander graph (i.e., $\lambda_2(G) \ll \lambda_1(G)$),
H = any d-regular bipartite graph.

Then:

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$\mathrm{vc}(H) = n/2$	by bipartition
$G \equiv^R H$	by regularity
$G \equiv_2^C H$	by Tinhofer's Theorem

Tight in two ways:

$G \not\equiv_3^C H$	bipartiteness is C^3-definable,
$G \equiv_2^C H \Longrightarrow \mathrm{vc}(G) \leq 2\mathrm{vc}(H)$	[AA-Dawar 2018]

A different weak (harder) case: $k = \Omega(n)$ but gap $= 1.08$

Theorem [AA-Dawar 2018]

There exist graphs G_n and H_n such that

1. $G_n \equiv^C_{\Omega(n)} H_n$
2. $\mathrm{vc}(G_n) \geq 1.08 \cdot \mathrm{vc}(H_n)$

Part V

PROOF INGREDIENTS

1/3: Locally consistent systems of linear equations

Ingredient 1: A linear system $Ax = b$ over $\mathbb{F}_2$ where:

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Probabilistic construction:

1. set $m = cn$ for a large constant $c = c(\epsilon)$
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Half-deterministic construction:

1. set $m = cn$ for a large constrant $c = c(\epsilon)$
2. let A be incidence matrix of bipartite expander
3. choose b uniformly at random in $\mathbb{F}_2^n$.

2/3: Indistinguishable systems of linear equations

Ingredient 2: A pair of linear systems S_0 and S_1 over $\mathbb{F}_2$ where:

2/3: Indistinguishable systems of linear equations

Ingredient 2: A pair of linear systems S_0 and S_1 over $\mathbb{F}_2$ where:

1. $S_0 \equiv^{C}_{\Omega(n)} S_1$

2/3: Indistinguishable systems of linear equations

Ingredient 2: A pair of linear systems S_0 and S_1 over $\mathbb{F}_2$ where:

1. $S_0 \equiv^{C}_{\Omega(n)} S_1$
2. every candidate solution for S_0 satisfies at most $\frac{3}{4}$ equations

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1. $S_0 \equiv^{C}_{\Omega(n)} S_1$
2. every candidate solution for S_0 satisfies at most $\frac{3}{4}$ equations
3. some solution solution exists for S_1

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Construction of S_0:

1. start with $Ax = b$ from previous section
2. duplicate each variable $x \mapsto (x^{(0)}, x^{(1)})$
3. replace each equation $x_i + x_j + x_k = b$ by 8 equations

$$x_i^{(u)} + x_j^{(v)} + x_k^{(w)} = b + u + v + w$$

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Construction of S_1:

1. same but start with $Ax = 0$ (the homogeneous system)

3/3: Reduction to vertex cover

Ingredient 3: A pair of graphs G_0 and G_1 where:

1. $G_0 \equiv^C_{\Omega(n)} G_1$
2. $\mathrm{vc}(G_0) \geq 26m$
3. $\mathrm{vc}(G_1) \leq 24m$

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Construction:

a standard reduction from $\mathbb{F}_2$-SAT to vertex cover

The CFI Construction

Theorem [Cai-Fürer-Immerman 92]:

There exists graphs G_n and H_n with n vertices such that

$$G_n \equiv^C_{\Omega(n)} H_n \quad \text{yet} \quad G_n \ncong H_n.$$

CFI construction

1. Start with a 3-regular graph G without $o(n)$-separators.

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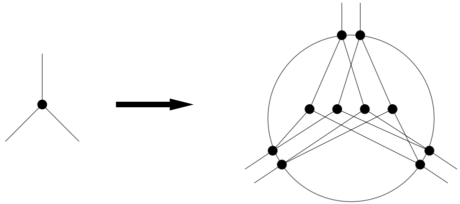

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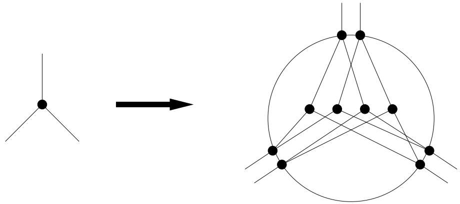
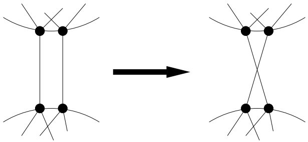


3. Let G_n be the result and let $H_n = G_n +$ “one flip”.



Part VI

CONCLUDING REMARKS

Open Problem 1

$$\sup_G \frac{\mathrm{vc}(G)}{\mathrm{sdp}^4\mathrm{fvc}(G)} > 1.36?$$

Open Problem 2

find strongly regular graphs G and H with same parameters so that $\mathrm{vc}(G) \geq (2-\epsilon)\mathrm{vc}(H)$.

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