Logical constants and logical consequence

Dag Westerståhl

Stockholm University

Tsinghua University, Beijing

(joint work with Denis Bonnay, Université Paris Nanterre, ENS)

Logic Colloquium '18

Udine

July 23 – 28, 2018

Overview

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

A model is a domain plus an interpretation of non-logical symbols.

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

A model is a domain plus an interpretation of non-logical symbols.

So, on which grounds is that choice made?

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

A model is a domain plus an interpretation of non-logical symbols.

So, on which grounds is that choice made?

Usually on pragmatic grounds: we pick a concept of model (a set of logical symbols) that works for our purposes.

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

A model is a domain plus an interpretation of non-logical symbols.

So, on which grounds is that choice made?

Usually on pragmatic grounds: we pick a concept of model (a set of logical symbols) that works for our purposes.

One would like to know something more about why certain choices work, and others not.

Why care about logical constants?

The answer is obvious: logical consequence depends on a choice of logical constants.

The modern notion of a model obscures this dependence somewhat.

A model is a domain plus an interpretation of non-logical symbols.

So, on which grounds is that choice made?

Usually on pragmatic grounds: we pick a concept of model (a set of logical symbols) that works for our purposes.

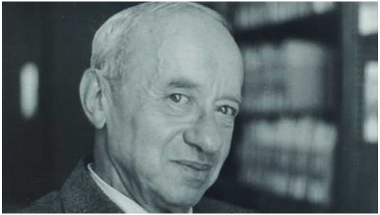
One would like to know something more about why certain choices work, and others not.

To begin, we observe that (sets of) constants and consequence relations (not necessarily logical) are dual notions.

Bolzano and Tarski

In 1837 Bolzano explicated the classical idea of logical consequence as preservation of truth under replacement of non-logical symbols.

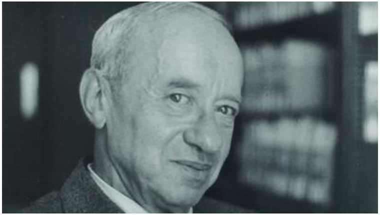
Bolzano and Tarski



In 1837 Bolzano explicated the classical idea of logical consequence as preservation of truth under replacement of non-logical symbols.

99 years later, Tarski explicated it as preservation of truth under reinterpretation of non-logical symbols.

Bolzano and Tarski

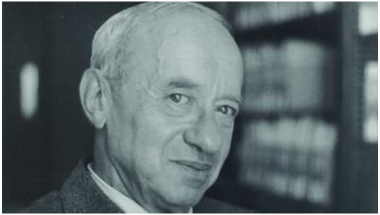


In 1837 Bolzano explicated the classical idea of logical consequence as preservation of truth under replacement of non-logical symbols.

99 years later, Tarski explicated it as preservation of truth under reinterpretation of non-logical symbols.

Which are these symbols you cannot replace/reinterpret?

Bolzano and Tarski



In 1837 Bolzano explicated the classical idea of logical consequence as preservation of truth under replacement of non-logical symbols.

99 years later, Tarski explicated it as preservation of truth under reinterpretation of non-logical symbols.

Which are these symbols you cannot replace/reinterpret?

Both Bolzano and Tarski emphasized the importance of this question, and that they had no answer.

Invariance

A commonly used criterion of logicality today is permutation invariance (suggested by Tarski 30 years after the 1936 paper), or, more generally, isomorphism invariance.

Invariance

A commonly used criterion of logicality today is permutation invariance (suggested by Tarski 30 years after the 1936 paper), or, more generally, isomorphism invariance.

Arguments: (a) logic is the most general science; (b) logic is topic neutral.

Invariance

A commonly used criterion of logicality today is permutation invariance (suggested by Tarski 30 years after the 1936 paper), or, more generally, isomorphism invariance.

Arguments: (a) logic is the most general science; (b) logic is topic neutral.

This has been much debated; e.g. Sher (1991), McGee (1996), Feferman (1999, 2010, 2015), Bonnay (2008).

Invariance

A commonly used criterion of logicality today is permutation invariance (suggested by Tarski 30 years after the 1936 paper), or, more generally, isomorphism invariance.

Arguments: (a) logic is the most general science; (b) logic is topic neutral.

This has been much debated; e.g. Sher (1991), McGee (1996), Feferman (1999, 2010, 2015), Bonnay (2008).

Most logicians now seem to agree that isomorphism invariance is a necessary condition for logicality, but few think it is sufficient.

Invariance

A commonly used criterion of logicality today is permutation invariance (suggested by Tarski 30 years after the 1936 paper), or, more generally, isomorphism invariance.

Arguments: (a) logic is the most general science; (b) logic is topic neutral.

This has been much debated; e.g. Sher (1991), McGee (1996), Feferman (1999, 2010, 2015), Bonnay (2008).

Most logicians now seem to agree that isomorphism invariance is a necessary condition for logicality, but few think it is sufficient.

So it makes sense to look for additional criteria of logicality.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between ‘logical’ and ‘non-logical’ symbols!

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between 'logical' and 'non-logical' symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between 'logical' and 'non-logical' symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

A compositional assignment of semantic values extends I to all expressions.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between ‘logical’ and ‘non-logical’ symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

A compositional assignment of semantic values extends I to all expressions.

In particular, this yields a truth relation $\models_L \subseteq Int_L \times Sent_L$.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between ‘logical’ and ‘non-logical’ symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

A compositional assignment of semantic values extends I to all expressions.

In particular, this yields a truth relation $\models_L \subseteq Int_L \times Sent_L$.

There is a standard interpretation $I_L \in Int_L$.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between 'logical' and 'non-logical' symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

A compositional assignment of semantic values extends I to all expressions.

In particular, this yields a truth relation $\models_L \subseteq Int_L \times Sent_L$.

There is a standard interpretation $I_L \in Int_L$.

- L is 'Tarskian' since it allows (re)interpretation of symbols.

(Tarskian) interpreted languages

SYNTAX: L generates sentences and possibly other expressions from a set of primitive symbols (also variable-binding operators).

No distinction between 'logical' and 'non-logical' symbols!

SEMANTICS: A class Int_L of interpretations: $I \in Int_L$ maps each $u \in Symb_L$ to a suitable semantic value $I(u)$.

A compositional assignment of semantic values extends I to all expressions.

In particular, this yields a truth relation $\models_L \subseteq Int_L \times Sent_L$.

There is a standard interpretation $I_L \in Int_L$.

- L is 'Tarskian' since it allows (re)interpretation of symbols.
- L is 'interpreted' since there is a standard interpretation.

The Bolzano-Tarski map

In effect, Bolzano and (less explicitly) Tarski map sets X of symbols to consequence relations:

The Bolzano-Tarski map

In effect, Bolzano and (less explicitly) Tarski map sets X of symbols to consequence relations:

(Bolzano) $\Gamma \Rightarrow_X^B \varphi$ iff for every replacement ρ of symbols which is the identity on X, $I_L \models \Gamma[\rho]$ implies $I_L \models \varphi[\rho]$.

The Bolzano-Tarski map

In effect, Bolzano and (less explicitly) Tarski map sets X of symbols to consequence relations:

(Bolzano) $\Gamma \Rightarrow^B_X \varphi$ iff for every replacement ρ of symbols which is the identity on X, $I_L \models \Gamma[\rho]$ implies $I_L \models \varphi[\rho]$.

(Tarski) $\Gamma \Rightarrow_X \varphi$ iff for every J s.t. $J =_X I_L$, $J \models \Gamma$ implies $J \models \varphi$.

($I =_X I'$ means that I and I' agree on X.)

The Bolzano-Tarski map

In effect, Bolzano and (less explicitly) Tarski map sets X of symbols to consequence relations:

(Bolzano) $\Gamma \Rightarrow^B_X \varphi$ iff for every replacement ρ of symbols which is the identity on X, $I_L \models \Gamma[\rho]$ implies $I_L \models \varphi[\rho]$.

(Tarski) $\Gamma \Rightarrow_X \varphi$ iff for every J s.t. $J =_X I_L$, $J \models \Gamma$ implies $J \models \varphi$.

($I =_X I'$ means that I and I' agree on X.)

Depending on the resources of L, $\Rightarrow^B_X$ and $\Rightarrow_X$ may differ.

The Bolzano-Tarski map

In effect, Bolzano and (less explicitly) Tarski map sets X of symbols to consequence relations:

(Bolzano) $\Gamma \Rightarrow^B_X \varphi$ iff for every replacement ρ of symbols which is the identity on X, $I_L \models \Gamma[\rho]$ implies $I_L \models \varphi[\rho]$.

(Tarski) $\Gamma \Rightarrow_X \varphi$ iff for every J s.t. $J =_X I_L$, $J \models \Gamma$ implies $J \models \varphi$.

($I =_X I'$ means that I and I' agree on X.)

Depending on the resources of L, $\Rightarrow^B_X$ and $\Rightarrow_X$ may differ.

Bonnay and W-hl (2012) show that if L admits expansions (in a precise sense), the two coincide.

The semantic concept of consequence

Thus, we take the map

$$X \mapsto \Rightarrow_X$$

to be the semantic concept of consequence.

The semantic concept of consequence

Thus, we take the map

$$X \mapsto \Rightarrow_X$$

to be the semantic concept of consequence.

NB X can be any subset of $Symb_L$.

The semantic concept of consequence

Thus, we take the map

$$X \mapsto \Rightarrow_X$$

to be the semantic concept of consequence.

NB X can be any subset of $Symb_L$.

The map yields — in Bolzano's spirit — more or less intuitive notions of consequence depending on the choice of X.

The semantic concept of consequence

Thus, we take the map

$$X \mapsto \Rightarrow_X$$

to be the semantic concept of consequence.

NB X can be any subset of $Symb_L$.

The map yields — in Bolzano's spirit — more or less intuitive notions of consequence depending on the choice of X.

It says nothing specifically about logical consequence.

A very simple idea of constancy

Idea: a word/symbol is a constant iff it is essential to inference.

A very simple idea of constancy

Idea: a word/symbol is a constant iff it is essential to inference.

Every shark bites
Leon is a shark
Leon bites

OK

A very simple idea of constancy

Idea: a word/symbol is a constant iff it is essential to inference.

Every shark bites	Every dog bites
Leon is a shark	Leon is a dog
Leon bites	Leon bites
OK	OK

A very simple idea of constancy

Idea: a word/symbol is a constant iff it is essential to inference.

Every shark bites	Every dog bites	Some shark bites
Leon is a shark	Leon is a dog	Leon is a shark
Leon bites	Leon bites	Leon bites
OK	OK	not OK!

A very simple idea of constancy

Idea: a word/symbol is a constant iff it is essential to inference.

Every shark bites	Every dog bites	Some shark bites
Leon is a shark	Leon is a dog	Leon is a shark
Leon bites	Leon bites	Leon bites
OK	OK	not OK!

So "shark" is not a constant (and similarly "bites", "dog"), but "every" is.

A map in the other direction

Let *CONS* be the set of consequence relations over L.

A map in the other direction

Let *CONS* be the set of consequence relations over L.

(Subsets of $\mathcal{P}(Sent_L) \times Sent_L$ satisfying some closure conditions.)

A map in the other direction

Let $CONS$ be the set of consequence relations over L.

(Subsets of $\mathcal{P}(Sent_L) \times Sent_L$ satisfying some closure conditions.)

$BTCONS$ is the subset of Bolzano-Tarski consequence relations.

A map in the other direction

Let *CONS* be the set of consequence relations over L.

(Subsets of $\mathcal{P}(Sent_L) \times Sent_L$ satisfying some closure conditions.)

BTCONS is the subset of Bolzano-Tarski consequence relations.

(The range of the map $\Rightarrow_{\_}$.)

A map in the other direction

Let *CONS* be the set of consequence relations over L.

(Subsets of $\mathcal{P}(Sent_L) \times Sent_L$ satisfying some closure conditions.)

BTCONS is the subset of Bolzano-Tarski consequence relations.

(The range of the map $\Rightarrow_{\_}$.)

From any $\vdash \in CONS$, we can extract its constants as follows:

Definition

$Const(\vdash) = \{u \in Symb_L : \exists\, \Gamma, \varphi, u'$ such that $\Gamma \vdash \varphi$ but $\Gamma[u'/u] \nvdash \varphi[u'/u]\}$

A map in the other direction

Let *CONS* be the set of consequence relations over L.

(Subsets of $\mathcal{P}(Sent_L) \times Sent_L$ satisfying some closure conditions.)

BTCONS is the subset of Bolzano-Tarski consequence relations.

(The range of the map $\Rightarrow_{\_}$.)

From any $\vdash \in CONS$, we can extract its constants as follows:

Definition

$Const(\vdash) = \{u \in Symb_L : \exists \Gamma, \varphi, u' \text{such that } \Gamma \vdash \varphi \text{ but } \Gamma[u'/u] \nvdash \varphi[u'/u]\}$

This is a syntactic way to extract constants from any consequence relation, building on an extremely simple but intuitive idea.

A Galois duality

Theorem (Bonnay and W-hl, 2012)

Restricting attention to compact consequence relations:

(a) *The pair of functions* $(Const, \Rightarrow_{\_})$ *is a monotone Galois connection from* $(BTCONS, \subseteq)$ *to* $(\mathcal{P}(Symb), \subseteq)$*. I.e. for* $\vdash \in BTCONS$ *and* $X \subseteq Symb$*,*

$$\vdash \; \subseteq \; \Rightarrow_X \;\; \textit{iff} \;\; Const(\vdash) \subseteq X$$

A Galois duality

Theorem (Bonnay and W-hl, 2012)

Restricting attention to compact consequence relations:

(a) *The pair of functions* $(Const, \Rightarrow_{\_})$ *is a monotone Galois connection from* $(BTCONS, \subseteq)$ *to* $(\mathcal{P}(Symb), \subseteq)$*. I.e. for* $\vdash \in BTCONS$ *and* $X \subseteq Symb$*,*

$$\vdash \; \subseteq \; \Rightarrow_X \;\; \text{iff} \;\; Const(\vdash) \subseteq X$$

(b)
$$\Rightarrow_X \; = \; \Rightarrow_{Const(\Rightarrow_X)}$$

in fact, $Const(\Rightarrow_X)$ *is minimal among sets of symbols* Y *s.t.* $\Rightarrow_Y = \Rightarrow_X$.

A Galois duality

Theorem (Bonnay and W-hl, 2012)

Restricting attention to compact consequence relations:

(a) *The pair of functions* $(Const, \Rightarrow_{\_})$ *is a monotone Galois connection from* $(BTCONS, \subseteq)$ *to* $(\mathcal{P}(Symb), \subseteq)$*. I.e. for* $\vdash \in BTCONS$ *and* $X \subseteq Symb$*,*

$$\vdash \subseteq \Rightarrow_X \text{ iff } Const(\vdash) \subseteq X$$

(b)
$$\Rightarrow_X = \Rightarrow_{Const(\Rightarrow_X)}$$

in fact, $Const(\Rightarrow_X)$ *is minimal among sets of symbols* Y *s.t.* $\Rightarrow_Y = \Rightarrow_X$.

NB This does not extend to all of *CONS*: in general the map *Const* is not monotone.

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Let INT consist of those classes of interpretations that contain the standard interpretation I_L.

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Let *INT* consist of those classes of interpretations that contain the standard interpretation I_L. For $\vdash \in CONS$ and $K \in INT$, define

$$Val(\vdash) = \{I \in Int_L : \text{ if } \Gamma \vdash \varphi \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Let *INT* consist of those classes of interpretations that contain the standard interpretation I_L. For $\vdash \in CONS$ and $K \in INT$, define

$$Val(\vdash) = \{I \in Int_L: \text{ if } \Gamma \vdash \varphi \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$

$$Log(K) = \{(\Gamma, \varphi): \text{ if } I \in K \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Let *INT* consist of those classes of interpretations that contain the standard interpretation I_L. For $\vdash \in CONS$ and $K \in INT$, define

$$Val(\vdash) = \{I \in Int_L : \text{ if } \Gamma \vdash \varphi \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$
$$Log(K) = \{(\Gamma, \varphi) : \text{ if } I \in K \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$

Fact

(Val, Log) is an antitone Galois conn. between $(INT, \subseteq)$ and $(CONS, \subseteq)$:

$$K \subseteq Val(\vdash) \textit{ iff } \vdash \subseteq Log(K)$$

Towards a semantic extraction method

Although simple and intuitive, syntactic extraction with *Const* may pick other symbols than logical ones.

Semantic methods of extraction allow for finer distinctions.

Let *INT* consist of those classes of interpretations that contain the standard interpretation I_L. For $\vdash \in CONS$ and $K \in INT$, define

$$Val(\vdash) = \{I \in Int_L : \text{ if } \Gamma \vdash \varphi \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$
$$Log(K) = \{(\Gamma, \varphi) : \text{ if } I \in K \text{ and } I \models \Gamma \text{ then } I \models \varphi\}$$

Fact

(Val, Log) *is an antitone Galois conn. between* $(INT, \subseteq)$ *and* $(CONS, \subseteq)$*:*

$$K \subseteq Val(\vdash) \ \textit{iff} \ \vdash \subseteq Log(K)$$

(Compare first-order logic: $K \subseteq Mod(\Psi)$ iff $\Psi \subseteq Th(K)$.)

Semantic extraction of constants

Idea: ‘extract’ symbols from $\mathit{Val}(\vdash)$, rather than from $\vdash$ directly.

Semantic extraction of constants

Idea: 'extract' symbols from $Val(\vdash)$, rather than from $\vdash$ directly.

Here is one way:

call u uniform in K iff for all $I, I' \in K,\ I(u) = I'(u)$.

Semantic extraction of constants

Idea: 'extract' symbols from $Val(\vdash)$, rather than from $\vdash$ directly.

Here is one way:

call u uniform in K iff for all $I, I' \in K$, $I(u) = I'(u)$.

If $K \in INT$, this amounts to $I(u) = I_L(u)$ for all $I \in K$.

Semantic extraction of constants

Idea: 'extract' symbols from $Val(\vdash)$, rather than from $\vdash$ directly.

Here is one way:

call u uniform in K iff for all $I, I' \in K$, $I(u) = I'(u)$.

If $K \in INT$, this amounts to $I(u) = I_L(u)$ for all $I \in K$.

Definition

$Unif(K) = \{u \in Symb\colon u \text{ is uniform in } K\}$.

Semantic extraction of constants

Idea: 'extract' symbols from $Val(\vdash)$, rather than from $\vdash$ directly.

Here is one way:

call u uniform in K iff for all $I, I' \in K$, $I(u) = I'(u)$.

If $K \in INT$, this amounts to $I(u) = I_L(u)$ for all $I \in K$.

Definition

$Unif(K) = \{u \in Symb\colon u \text{ is uniform in } K\}$.

In the other direction:

$Stand(X) = \{I\colon I =_X I_L\}$

Semantic extraction of constants

Idea: 'extract' symbols from $Val(\vdash)$, rather than from $\vdash$ directly.

Here is one way:

call u uniform in K iff for all $I, I' \in K$, $I(u) = I'(u)$.

If $K \in INT$, this amounts to $I(u) = I_L(u)$ for all $I \in K$.

Definition

$Unif(K) = \{u \in Symb\colon u \text{ is uniform in } K\}$.

In the other direction:

$Stand(X) = \{I : I =_X I_L\}$

Note:

$$\Rightarrow_X \;=\; Log(Stand(X))$$

Another Galois connection

Recall:

$$Unif(K) = \{u \in Symb_L : I(u) = I_L(u) \text{ for all } I \in K\}$$
$$Stand(X) = \{I \in Int_L : I =_X I_L\}$$

Another Galois connection

Recall:

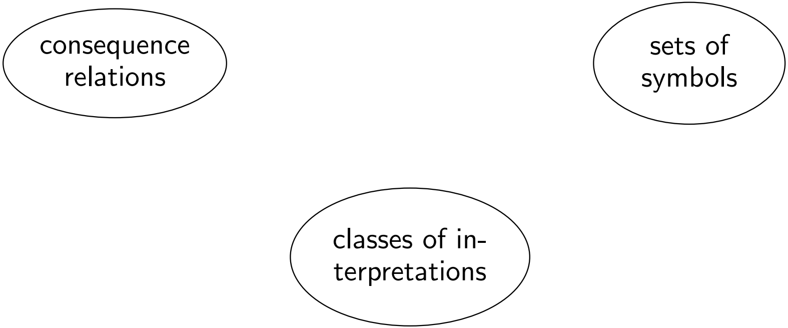
$$Unif(K) = \{u \in Symb_L : I(u) = I_L(u) \text{ for all } I \in K\}$$
$$Stand(X) = \{I \in Int_L : I =_X I_L\}$$

Proposition

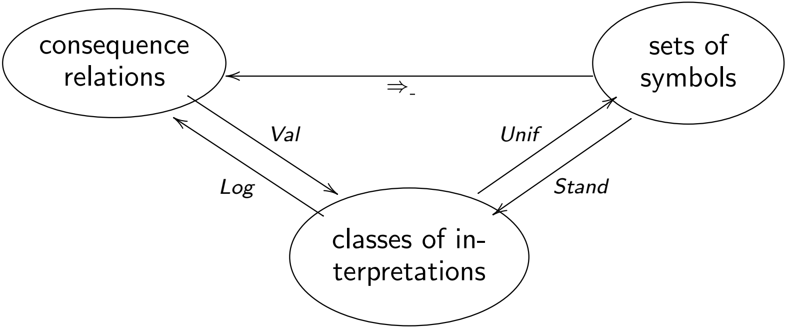
$(Unif, Stand)$ is an antitone Galois connection between $(INT, \subseteq)$ and $(\mathcal{P}(Symb), \subseteq)$:

$$K \subseteq Stand(X) \;\; iff \;\; X \subseteq Unif(K)$$

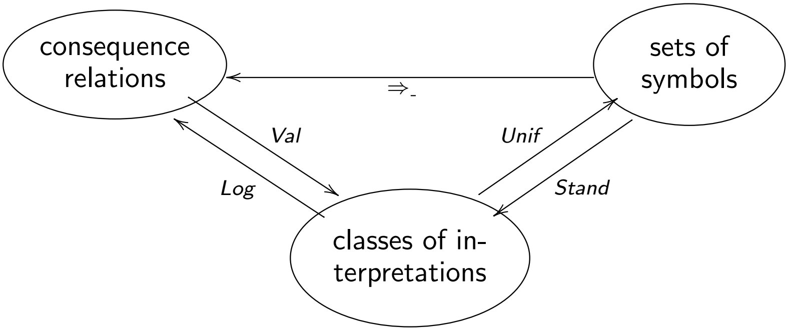
Semantic extraction of constants



Semantic extraction of constants



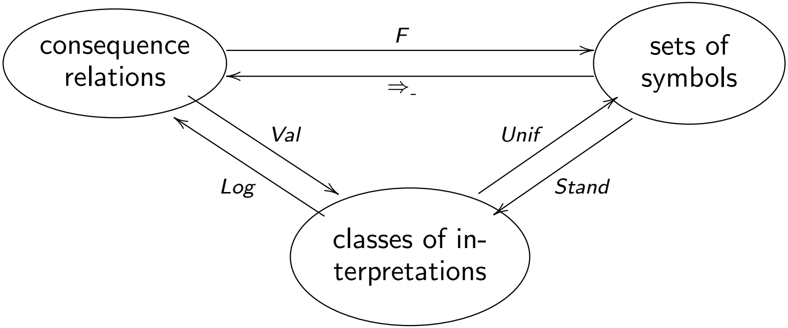
Semantic extraction of constants



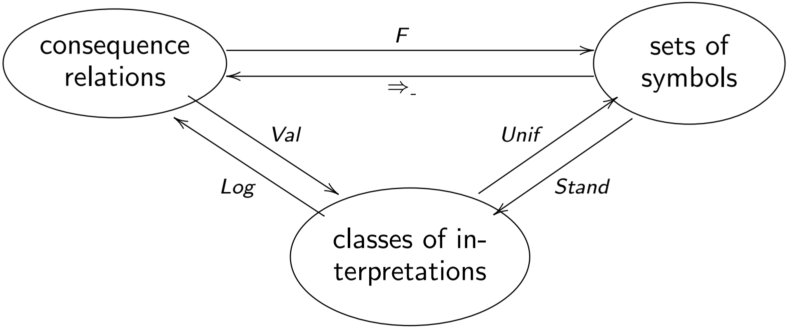
$K \subseteq Val(\vdash)$ iff $\vdash \subseteq Log(K)$

$K \subseteq Stand(X)$ iff $X \subseteq Unif(K)$

Note 1: antitone Galois connections don't compose

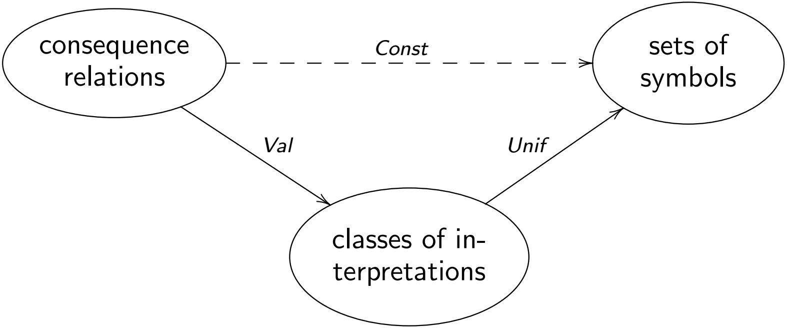


Note 1: antitone Galois connections don't compose



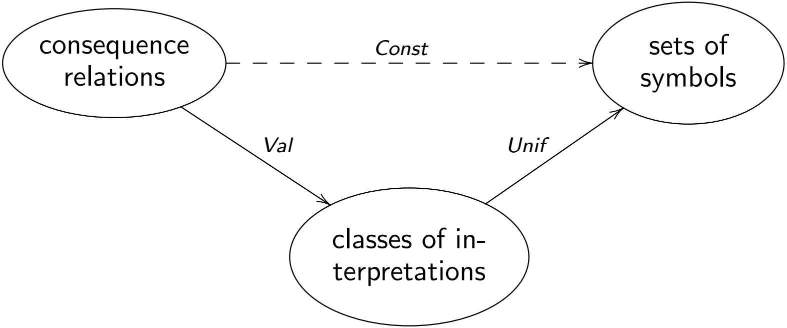
Although $F = Unif \circ Val$ and $\Rightarrow_{\_} = Log \circ Stand$ are monotone functions, $(F, \Rightarrow_{\_})$ is not a Galois connection.

Note 2: a semantic version of *Const*



Unif (*Val*($\vdash$) is usually a proper subset of the syntactic *Const*($\vdash$).

Note 2: a semantic version of *Const*



$Unif(Val(\vdash)$ is usually a proper subset of the syntactic $Const(\vdash)$.

One can define another semantic extraction function H, with $Unif(K) \subseteq H(K)$ for all K, such that when $\vdash = Log(K)$,

$$H(Val(\vdash) = Const(\vdash)$$

Unif and analytic inference

Let $\vdash$ be a pretheoretic notion of consequence, validating both (1) and (2):

(1) Phil is good-looking *and* he is a student
Hence: Phil is a student.

(2) Phil is good-looking *and* he is a *student*
Hence: Phil is not a *dog*.

Unif and analytic inference

Let $\vdash$ be a pretheoretic notion of consequence, validating both (1) and (2):

(1) Phil is good-looking *and* he is a student
Hence: Phil is a student.

(2) Phil is good-looking *and* he is a *student*
Hence: Phil is not a *dog*.

Intuitively, (1) is logically valid, but (2) is only analytically valid.

Unif and analytic inference

Let $\vdash$ be a pretheoretic notion of consequence, validating both (1) and (2):

(1) Phil is good-looking *and* he is a student
Hence: Phil is a student.

(2) Phil is good-looking *and* he is a *student*
Hence: Phil is not a *dog*.

Intuitively, (1) is logically valid, but (2) is only analytically valid.

Unif is sensitive to this difference, in contrast with *Const*.

Unif and analytic inference

Let $\vdash$ be a pretheoretic notion of consequence, validating both (1) and (2):

(1) Phil is good-looking *and* he is a student
Hence: Phil is a student.

(2) Phil is good-looking *and* he is a *student*
Hence: Phil is not a *dog*.

Intuitively, (1) is logically valid, but (2) is only analytically valid.

Unif is sensitive to this difference, in contrast with *Const*. Also:

Fact

$\Rightarrow_{Unif(K)} \subseteq Log(K)$

Unif and analytic inference

Let $\vdash$ be a pretheoretic notion of consequence, validating both (1) and (2):

(1) Phil is good-looking *and* he is a student
Hence: Phil is a student.

(2) Phil is good-looking *and* he is a *student*
Hence: Phil is not a *dog*.

Intuitively, (1) is logically valid, but (2) is only analytically valid.

Unif is sensitive to this difference, in contrast with *Const*. Also:

Fact

$\Rightarrow_{Unif(K)} \subseteq Log(K)$

Unif(K) can sometimes isolate the 'logical part' of a consequence relation $Log(K)$.

U and analytic inference: an example

Let I_L describe a domain M, with L using ordinary atomic sentences and propositional logic, and having a name for each element of M:

$Symb_L = \{\neg, \wedge, \rightarrow, student, human, dog, \ldots\} \cup \{\bar{a} \colon a \in M\}$.

U and analytic inference: an example

Let I_L describe a domain M, with L using ordinary atomic sentences and propositional logic, and having a name for each element of M:

$$Symb_L = \{\neg, \wedge, \rightarrow, student, human, dog, \ldots\} \cup \{\bar{a} \colon a \in M\}.$$

Consider meaning postulates such as

(ψ_a) $student(\bar{a}) \rightarrow human(\bar{a})$, for $a \in M$

and define a consequence relation $\vdash$ by

$$\Gamma \vdash \varphi \text{ iff } \Gamma \cup \{\psi_a \colon a \in M\} \models^{PL} \varphi$$

U and analytic inference: an example

Let I_L describe a domain M, with L using ordinary atomic sentences and propositional logic, and having a name for each element of M:

$Symb_L = \{\neg, \wedge, \rightarrow, student, human, dog, \dots\} \cup \{\bar{a} \colon a \in M\}$.

Consider meaning postulates such as

(ψ_a) $student(\bar{a}) \rightarrow human(\bar{a})$, for $a \in M$

and define a consequence relation $\vdash$ by

$$\Gamma \vdash \varphi \text{ iff } \Gamma \cup \{\psi_a \colon a \in M\} \models^{PL} \varphi$$

Fact

(a) $Const(\vdash) = \{\neg, \wedge, \rightarrow, student, human, dog, \dots\}$

(b) $Unif(Val(\vdash)) = \{\neg, \wedge, \rightarrow\}$

(c) $\Rightarrow_{Unif(Val(\vdash))} = \models^{PL}$

Syntax matching semantics

This suggest that $Unif(Val(\vdash))$ is a way of identifying the logical constants with their standard meaning.

Syntax matching semantics

This suggest that $\mathit{Unif}(\mathit{Val}(\vdash))$ is a way of identifying the logical constants with their standard meaning.

In particular if we start from familiar logical consequence relations.

Syntax matching semantics

This suggest that $Unif(Val(\vdash))$ is a way of identifying the logical constants with their standard meaning.

In particular if we start from familiar logical consequence relations.

Such consequence relations need not be Bolzano-Tarskian, but can be defined syntactically by rules.

Syntax matching semantics

This suggest that $Unif(Val(\vdash))$ is a way of identifying the logical constants with their standard meaning.

In particular if we start from familiar logical consequence relations.

Such consequence relations need not be Bolzano-Tarskian, but can be defined syntactically by rules.

When syntactically defined consequence coincides with Bolzano-Tarski consequence wrt some natural semantics, we have soundness and completeness theorems: syntax matches semantics.

Semantics matching syntax

One can argue that semantics should also match syntax.

Semantics matching syntax

One can argue that semantics should also match syntax.

The meaning of the logical vocabulary doesn't come out of nowhere; it should somehow derive from, or at least be strongly tied to, the relation of logical consequence.

Semantics matching syntax

One can argue that semantics should also match syntax.

The meaning of the logical vocabulary doesn't come out of nowhere; it should somehow derive from, or at least be strongly tied to, the relation of logical consequence.

This may be established by categoricity theorems, showing that the standard interpretation is the unique one guaranteeing correctness.

Semantics matching syntax

One can argue that semantics should also match syntax.

The meaning of the logical vocabulary doesn't come out of nowhere; it should somehow derive from, or at least be strongly tied to, the relation of logical consequence.

This may be established by categoricity theorems, showing that the standard interpretation is the unique one guaranteeing correctness.

From now on we focus on logical consequence and logical constants, applying the semantic 'extraction method' just described.

Carnap's *Formalization in Logic* (1943)

Early on, Carnap considered such categoricity results as core metalogical results, on a par with soundness and completeness.

Carnap's *Formalization in Logic* (1943)

Early on, Carnap considered such categoricity results as core metalogical results, on a par with soundness and completeness.

These ideas were somehow forgotten along the way.

Carnap's *Formalization in Logic* (1943)

Early on, Carnap considered such categoricity results as core metalogical results, on a par with soundness and completeness.

These ideas were somehow forgotten along the way.

Carnap worried that even the classical meaning of the propositional connectives might not be fixed by classical consequence, since he found valuations consistent with $\models^{PL}$ that gave non-standard truth tables for $\vee$, $\rightarrow$, and $\neg$;

Carnap's *Formalization in Logic* (1943)

Early on, Carnap considered such categoricity results as core metalogical results, on a par with soundness and completeness.

These ideas were somehow forgotten along the way.

Carnap worried that even the classical meaning of the propositional connectives might not be fixed by classical consequence, since he found valuations consistent with $\models^{PL}$ that gave non-standard truth tables for $\vee$, $\rightarrow$, and $\neg$; e.g. the valuation that assigns all and only tautologies the value True.

Carnap's Question

Let us call this Carnap's Question:

(CQ) To what extent does a logical consequence relation $\vdash$ among sentences in a formal language L determine the meanings of the logical L-symbols?

Carnap's Question

Let us call this Carnap's Question:

(CQ) To what extent does a logical consequence relation $\vdash$ among sentences in a formal language L determine the meanings of the logical L-symbols?

In other words:

> Under what circumstances does $Unif(Val(\vdash))$ single out the standard logical constants in L?

Carnap's Question

Let us call this Carnap's Question:

(CQ) To what extent does a logical consequence relation $\vdash$ among sentences in a formal language L determine the meanings of the logical L-symbols?

In other words:

> Under what circumstances does $Unif(Val(\vdash))$ single out the standard logical constants in L?

For example: to what extent does standard first-order consequence $\models^{FO}$ fix the meanings of $\neg$, $\wedge$, and $\forall$?

Carnap's Question

Let us call this Carnap's Question:

(CQ) To what extent does a logical consequence relation $\vdash$ among sentences in a formal language L determine the meanings of the logical L-symbols?

In other words:

> Under what circumstances does $\mathit{Unif}(\mathit{Val}(\vdash))$ single out the standard logical constants in L?

For example: to what extent does standard first-order consequence $\models^{FO}$ fix the meanings of $\neg$, $\wedge$, and $\forall$?

Or: Does the consequence relation $\vdash^{IL}$ for intuitionistic propositional logic fix the intuitionistic meaning of $\vee$, $\wedge$, $\rightarrow$, $\bot$?

Carnap's Question

Let us call this Carnap's Question:

(CQ) To what extent does a logical consequence relation $\vdash$ among sentences in a formal language L determine the meanings of the logical L-symbols?

In other words:

> Under what circumstances does $Unif(Val(\vdash))$ single out the standard logical constants in L?

For example: to what extent does standard first-order consequence $\models^{FO}$ fix the meanings of $\neg$, $\wedge$, and $\forall$?

Or: Does the consequence relation $\vdash^{IL}$ for intuitionistic propositional logic fix the intuitionistic meaning of $\vee$, $\wedge$, $\rightarrow$, $\bot$?

This question can be asked for any logic.

Propositional logic

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ *consistent with* $\models^{PL}$ *are the standard truth functions (truth tables).*

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ *consistent with* $\models^{PL}$ *are the standard truth functions (truth tables).*

- Crucially, valuations are here assumed to be compositional.

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ *consistent with* $\models^{PL}$ *are the standard truth functions (truth tables).*

- Crucially, valuations are here assumed to be compositional.

  (Assigning True to all and only tautologies is not compositional!)

Propositional logic

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ consistent with $\models^{PL}$ are the standard truth functions (truth tables).

- Crucially, valuations are here assumed to be compositional.

  (Assigning True to all and only tautologies is not compositional!)
- If the semantic values of formulas are truth values, compositionality forces us to interpret connectives as functions from truth values to truth values, i.e. as truth functions.

Propositional logic

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ *consistent with* $\models^{PL}$ *are the standard truth functions (truth tables).*

- Crucially, valuations are here assumed to be compositional.

  (Assigning True to all and only tautologies is not compositional!)
- If the semantic values of formulas are truth values, compositionality forces us to interpret connectives as functions from truth values to truth values, i.e. as truth functions.
- $\therefore$ We have a complete answer to CQ for classical *PL*.

Propositional logic

The following fact is essentially in Carnap's book:

Fact

The only truth functions for $\neg$, $\vee$, $\wedge$, $\rightarrow$ consistent with $\models^{PL}$ are the standard truth functions (truth tables).

- Crucially, valuations are here assumed to be compositional.

  (Assigning True to all and only tautologies is not compositional!)
- If the semantic values of formulas are truth values, compositionality forces us to interpret connectives as functions from truth values to truth values, i.e. as truth functions.
- $\therefore$ We have a complete answer to CQ for classical PL.
- Carnap didn't see this fact as a solution — the idea of formal (compositional) semantics wasn't around in 1943!

First-order logic

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

First-order logic

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

The interpretation of $\forall$ is any type $\langle 1 \rangle$ generalized quantifier.

First-order logic

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

The interpretation of $\forall$ is any type $\langle 1 \rangle$ generalized quantifier.

There is a unique standard interpretation; in particular, on each domain M, $I_L(\forall) = \{M\}$.

First-order logic

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

The interpretation of $\forall$ is any type $\langle 1 \rangle$ generalized quantifier.

There is a unique standard interpretation; in particular, on each domain M, $I_L(\forall) = \{M\}$.

By the earlier result, the connectives are standard, so we only need to consider $\forall$ (and identity).

First-order logic

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

The interpretation of $\forall$ is any type $\langle 1 \rangle$ generalized quantifier.

There is a unique standard interpretation; in particular, on each domain M, $I_L(\forall) = \{M\}$.

By the earlier result, the connectives are standard, so we only need to consider $\forall$ (and identity).

So interpretations are essentially weak models: $(\mathcal{M}, \mathcal{Q})$, where $\mathcal{M}$ is an ordinary first-order model and $\mathcal{Q} \subseteq \mathcal{P}(M)$ interprets $\forall$.

We use standard first-order semantics, except that every symbol (including connectives and quantifiers) must be interpreted.

The interpretation of $\forall$ is any type $\langle 1 \rangle$ generalized quantifier.

There is a unique standard interpretation; in particular, on each domain M, $I_L(\forall) = \{M\}$.

By the earlier result, the connectives are standard, so we only need to consider $\forall$ (and identity).

So interpretations are essentially weak models: $(\mathcal{M}, \mathcal{Q})$, where $\mathcal{M}$ is an ordinary first-order model and $\mathcal{Q} \subseteq \mathcal{P}(M)$ interprets $\forall$.

It is easy to see that consistency with $\models^{FO}$ (i.e. I belonging to $Val(\models^{FO})$) forces $I(\forall) = \mathcal{Q}$ to be a filter.

Carnap's Question for *FO*

Theorem (Bonnay & W-hl 2016)

The interpretation I *over a domain* M *is consistent with* $\models^{FO}$ *iff* $I(\forall)$ *is a principal filter.*

Carnap's Question for *FO*

Theorem (Bonnay & W-hl 2016)

The interpretation I over a domain M is consistent with $\models^{FO}$ iff $I(\forall)$ is a principal filter. Moreover, if $I(\forall)$ is required to be permutation invariant, then $I(\forall) = \{M\}$, i.e. $\forall$ must have its standard interpretation.

Carnap's Question for *FO*

Theorem (Bonnay & W-hl 2016)

The interpretation I over a domain M is consistent with $\models^{FO}$ iff $I(\forall)$ is a principal filter. Moreover, if $I(\forall)$ is required to be permutation invariant, then $I(\forall) = \{M\}$, i.e. $\forall$ must have its standard interpretation. Also, in this case, $I(=)$ is standard identity.

Carnap's Question for *FO*

Theorem (Bonnay & W-hl 2016)

The interpretation I over a domain M is consistent with $\models^{FO}$ iff $I(\forall)$ is a principal filter. Moreover, if $I(\forall)$ is required to be permutation invariant, then $I(\forall) = \{M\}$, i.e. $\forall$ must have its standard interpretation. Also, in this case, $I(=)$ is standard identity.

The property of *FO* forcing the filter $\mathcal{Q} = I(\forall)$ to be principal is:

$$\models^{FO} \forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi$$

which says that for all $R \subseteq M^2$, and with $R_a = \{b \colon aRb\}$,

$$\{a \colon R_a \in \mathcal{Q}\} \in \mathcal{Q} \;\Leftrightarrow\; \{a \colon (R^{-1})_a \in \mathcal{Q}\} \in \mathcal{Q} \quad \text{(commutativity)}$$

Carnap's Question for FO

Theorem (Bonnay & W-hl 2016)

The interpretation I over a domain M is consistent with $\models^{FO}$ iff $I(\forall)$ is a principal filter. Moreover, if $I(\forall)$ is required to be permutation invariant, then $I(\forall) = \{M\}$, i.e. $\forall$ must have its standard interpretation. Also, in this case, $I(=)$ is standard identity.

The property of FO forcing the filter $\mathcal{Q} = I(\forall)$ to be principal is:

$$\models^{FO} \forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi$$

which says that for all $R \subseteq M^2$, and with $R_a = \{b\colon aRb\}$,

$$\{a\colon R_a \in \mathcal{Q}\} \in \mathcal{Q} \;\Leftrightarrow\; \{a\colon (R^{-1})_a \in \mathcal{Q}\} \in \mathcal{Q} \quad \text{(commutativity)}$$

Proof idea: Note that the Fréchet filter is non-commutative, and generalize to all non-principal filters.

Carnap's Question for *FO*, cont.

When $\mathcal{Q}$, interpreting $\forall$, is a principal filter generated by a set A, we have:

(1) $(\mathcal{M}, \mathcal{Q}) \models \forall x \varphi [f]$ iff for all $a \in A$, $(\mathcal{M}, \mathcal{Q}) \models \varphi [f(x/a)]$

Carnap's Question for *FO*, cont.

When $\mathcal{Q}$, interpreting $\forall$, is a principal filter generated by a set A, we have:

(1) $(\mathcal{M}, \mathcal{Q}) \models \forall x \varphi [f]$ iff for all $a \in A$, $(\mathcal{M}, \mathcal{Q}) \models \varphi [f(x/a)]$

That is, classical first-order consequence forces $\forall$ to mean restricted universal quantification.

Carnap's Question for *FO*, cont.

When $\mathcal{Q}$, interpreting $\forall$, is a principal filter generated by a set A, we have:

(1) $(\mathcal{M}, \mathcal{Q}) \models \forall x \varphi [f]$ iff for all $a \in A$, $(\mathcal{M}, \mathcal{Q}) \models \varphi [f(x/a)]$

That is, classical first-order consequence forces $\forall$ to mean restricted universal quantification.

We see exactly how the interpretation of $\forall$ is constrained by $\models^{FO}$.

Carnap's Question for *FO*, cont.

When $\mathcal{Q}$, interpreting $\forall$, is a principal filter generated by a set A, we have:

(1) $(\mathcal{M}, \mathcal{Q}) \models \forall x \varphi [f]$ iff for all $a \in A$, $(\mathcal{M}, \mathcal{Q}) \models \varphi [f(x/a)]$

That is, classical first-order consequence forces $\forall$ to mean restricted universal quantification.

We see exactly how the interpretation of $\forall$ is constrained by $\models^{FO}$.

Although it is not completely fixed, an additional reasonable invariance requirement does fix it.

Modal logic

(What we call) Carnap's Question can be asked for any logic.

Modal logic

(What we call) Carnap's Question can be asked for any logic.

The question and its answer are completely clear for first-order logic, but additional conceptual issues arise for some other logics.

Modal logic

(What we call) Carnap's Question can be asked for any logic.

The question and its answer are completely clear for first-order logic, but additional conceptual issues arise for some other logics.

Modal logic is an instructive example.

Modal logic

(What we call) Carnap's Question can be asked for any logic.

The question and its answer are completely clear for first-order logic, but additional conceptual issues arise for some other logics.

Modal logic is an instructive example.

Syntax is unproblematic: we consider the basic modal language

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$$

(other operators defined as usual).

Modal semantics: choices

But there are several issues concerning the semantics:

Modal semantics: choices

But there are several issues concerning the semantics:

- Should we use Kripke semantics, or topological semantics, or neighborhood semantics, or algebraic semantics, or ...?

Modal semantics: choices

But there are several issues concerning the semantics:

- Should we use Kripke semantics, or topological semantics, or neighborhood semantics, or algebraic semantics, or . . . ?
- Even with standard Kripke semantics, the interpretation of $\Box$ is not fixed, but parametric in the accessibility relation R.

Modal semantics: choices

But there are several issues concerning the semantics:

- Should we use Kripke semantics, or topological semantics, or neighborhood semantics, or algebraic semantics, or ...?
- Even with standard Kripke semantics, the interpretation of $\Box$ is not fixed, but parametric in the accessibility relation R.
- And what is the standard interpretation of $\Box$?

Modal semantics: choices

But there are several issues concerning the semantics:

- Should we use Kripke semantics, or topological semantics, or neighborhood semantics, or algebraic semantics, or ...?
- Even with standard Kripke semantics, the interpretation of $\Box$ is not fixed, but parametric in the accessibility relation R.
- And what is the standard interpretation of $\Box$?
- There is just one first-order consequence relation, but innumerable modal consequence relations (modal logics).

Modal semantics: choices

But there are several issues concerning the semantics:

- Should we use Kripke semantics, or topological semantics, or neighborhood semantics, or algebraic semantics, or ...?
- Even with standard Kripke semantics, the interpretation of $\Box$ is not fixed, but parametric in the accessibility relation R.
- And what is the standard interpretation of $\Box$?
- There is just one first-order consequence relation, but innumerable modal consequence relations (modal logics).
- It turns out that permutation invariance doesn't help.

Modal semantics is local

We approach this by focusing on the familiar local character of modal logic.

Modal semantics is local

We approach this by focusing on the familiar local character of modal logic.

Locality in this context seems to mean two things:

Modal semantics is local

We approach this by focusing on the familiar local character of modal logic.

Locality in this context seems to mean two things:

(i) Truth relativity: modal formulas are true at worlds (states, points, indices, . . .).

Modal semantics is local

We approach this by focusing on the familiar local character of modal logic.

Locality in this context seems to mean two things:

(i) Truth relativity: modal formulas are true at worlds (states, points, indices, ...).

(ii) Truth locality: only worlds 'near' w matter for the truth of φ at w.

Modal semantics is local

We approach this by focusing on the familiar local character of modal logic.

Locality in this context seems to mean two things:

(i) Truth relativity: modal formulas are true at worlds (states, points, indices, ...).

(ii) Truth locality: only worlds 'near' w matter for the truth of φ at w.

(i) leads immediately to possible worlds semantics:

Possible worlds semantics

Definition

A possible worlds semantics for modal logic is one where the semantic values of formulas are sets of worlds.

Possible worlds semantics

Definition

A possible worlds semantics for modal logic is one where the semantic values of formulas are sets of worlds.

By compositionality, this also means that connectives must be interpreted as functions from sets of worlds to sets of worlds.

Possible worlds semantics

Definition

A possible worlds semantics for modal logic is one where the semantic values of formulas are sets of worlds.

By compositionality, this also means that connectives must be interpreted as functions from sets of worlds to sets of worlds.

Thus, over a domain W:

$$[\![\varphi]\!]_{(I,V)} \subseteq W$$

(V is a valuation).

Possible worlds semantics

Definition

A possible worlds semantics for modal logic is one where the semantic values of formulas are sets of worlds.

By compositionality, this also means that connectives must be interpreted as functions from sets of worlds to sets of worlds.

Thus, over a domain W:

$$[\![\varphi]\!]_{(I,V)} \subseteq W$$

(V is a valuation).

NB This is not Kripke semantics: nothing about accessibility so far!

The usual connectives are standard

Definition

An interpretation I (over W) is consistent with a consequence relation $\vdash$ (that is, $I \in \mathit{Val}(\mathit{Unif}(\vdash))$) if:

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \bigcap_{\psi \in \Gamma} [\![\psi]\!]_{(I,V)} \subseteq [\![\varphi]\!]_{(I,V)} \text{ for all valuations } V$$

The usual connectives are standard

Definition

An interpretation I (over W) is consistent with a consequence relation $\vdash$ (that is, $I \in Val(Unif(\vdash))$) if:

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \bigcap_{\psi \in \Gamma} [\![\psi]\!]_{(I,V)} \subseteq [\![\varphi]\!]_{(I,V)} \text{ for all valuations } V$$

The usual propositional connectives have standard interpretations:

$$I(\neg)(X) = W - X \qquad I(\wedge)(X, Y) = X \cap Y$$

The usual connectives are standard

Definition

An interpretation I (over W) is consistent with a consequence relation $\vdash$ (that is, $I \in Val(Unif(\vdash))$) if:

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \bigcap_{\psi \in \Gamma} [\![\psi]\!]_{(I,V)} \subseteq [\![\varphi]\!]_{(I,V)} \text{ for all valuations } V$$

The usual propositional connectives have standard interpretations:

$$I(\neg)(X) = W - X \qquad I(\wedge)(X, Y) = X \cap Y$$

Definition

A modal logic is a set of sentences in the basic modal language containing all tautologies and closed under Modus Ponens and Uniform Substitution.

The usual connectives are standard

Definition

An interpretation I (over W) is consistent with a consequence relation $\vdash$ (that is, $I \in Val(Unif(\vdash))$) if:

$$\Gamma \vdash \varphi \quad \Rightarrow \quad \bigcap_{\psi \in \Gamma} [\![\psi]\!]_{(I,V)} \subseteq [\![\varphi]\!]_{(I,V)} \text{ for all valuations } V$$

The usual propositional connectives have standard interpretations:

$$I(\neg)(X) = W - X \qquad I(\wedge)(X, Y) = X \cap Y$$

Definition

A modal logic is a set of sentences in the basic modal language containing all tautologies and closed under Modus Ponens and Uniform Substitution.

Some good news:

Theorem (Bonnay & W-hl 2016)

Only the standard interpretation of $\neg$, $\wedge$ is consistent with modal logics.

Interpretations and truth

So we only need to be concerned about the interpretation of $\Box$.

Interpretations and truth

So we only need to be concerned about the interpretation of $\Box$.

Let's make this official:

Definition

A local interpretation is a pair (W, F), where $F\colon \mathcal{P}(W) \to \mathcal{P}(W)$.

Interpretations and truth

So we only need to be concerned about the interpretation of $\Box$.

Let's make this official:

Definition

A local interpretation is a pair (W, F), where $F\colon \mathcal{P}(W) \to \mathcal{P}(W)$.

With $\mathcal{M} = (W, F, V)$, the truth definition is:

- $[\![p]\!]_{\mathcal{M}} = V(p)$
- $[\![\neg\varphi]\!]_{\mathcal{M}} = W - [\![\varphi]\!]_{\mathcal{M}}$
- $[\![\varphi \wedge \psi]\!]_{\mathcal{M}} = [\![\varphi]\!]_{\mathcal{M}} \cap [\![\psi]\!]_{\mathcal{M}}$
- $[\![\Box\varphi]\!]_{\mathcal{M}} = F([\![\varphi]\!]_{\mathcal{M}})$

($\Box$ means whatever F says it means.)

Local interpretations are neighborhood frames

The (W, F) are in fact familiar objects: neighborhood frames, often presented as (W, N), where $N\colon W \to \mathcal{P}(\mathcal{P}(W))$.

Local interpretations are neighborhood frames

The (W, F) are in fact familiar objects: neighborhood frames, often presented as (W, N), where $N\colon W \to \mathcal{P}(\mathcal{P}(W))$.

That's just another way to describe the same object; set

$$N_F(w) = \{X \subseteq W : w \in F(X)\}$$

and vice versa.

Local interpretations are neighborhood frames

The (W, F) are in fact familiar objects: neighborhood frames, often presented as (W, N), where $N \colon W \to \mathcal{P}(\mathcal{P}(W))$.

That's just another way to describe the same object; set

$$N_F(w) = \{X \subseteq W : w \in F(X)\}$$

and vice versa.

For $\mathcal{M} = (W, N, V)$ we now have:

$\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$ ($[\![\varphi]\!]_{\mathcal{M}}$ is a 'neighborhood' of w)

Local interpretations are neighborhood frames

The (W, F) are in fact familiar objects: neighborhood frames, often presented as (W, N), where $N\colon W \to \mathcal{P}(\mathcal{P}(W))$.

That's just another way to describe the same object; set

$$N_F(w) = \{X \subseteq W : w \in F(X)\}$$

and vice versa.

For $\mathcal{M} = (W, N, V)$ we now have:

$\mathcal{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \in N(w)$ ($[\![\varphi]\!]_{\mathcal{M}}$ is a 'neighborhood' of w)

Kripke frames, and topological frames, are special cases of neighborhood frames.

Neighborhood semantics

Summing up:

The characteristic truth relativity of modal logic leads to possible worlds semantics,

Neighborhood semantics

Summing up:

The characteristic truth relativity of modal logic leads to possible worlds semantics,

which with compositionality and the result about $\neg$ and $\wedge$ allows us to identify (local) interpretations of the basic modal language with neighborhood frames.

Neighborhood semantics

Summing up:

The characteristic truth relativity of modal logic leads to possible worlds semantics,

which with compositionality and the result about $\neg$ and $\wedge$ allows us to identify (local) interpretations of the basic modal language with neighborhood frames.

We arrive at the familiar territory of neighborhood semantics, but from a new perspective.

Kripkean interpretations

Definition

(W, F) is Kripkean if there is $R \subseteq W^2$ s.t.

$$F(X) = \{w \in W : R_w \subseteq X\}$$

where $R_w = \{v : wRv\}$. I.e. $\mathcal{M}, w \models \Box\varphi$ iff $\forall v(wRv \Rightarrow \mathcal{M}, v \models \varphi)$.

Kripkean interpretations

Definition

(W, F) is Kripkean if there is $R \subseteq W^2$ s.t.

$$F(X) = \{w \in W : R_w \subseteq X\}$$

where $R_w = \{v : wRv\}$. I.e. $\mathcal{M}, w \models \Box\varphi$ iff $\forall v (wRv \Rightarrow \mathcal{M}, v \models \varphi)$.

For any neighborhood frame (W, F), define its potential accessibility relation Acc_F:

$$w \, Acc_F \, v \quad \text{iff} \quad v \in \bigcap N_F(w)$$

Kripkean interpretations

Definition

(W, F) is Kripkean if there is $R \subseteq W^2$ s.t.

$$F(X) = \{w \in W : R_w \subseteq X\}$$

where $R_w = \{v : wRv\}$. I.e. $\mathcal{M}, w \models \Box\varphi$ iff $\forall v(wRv \Rightarrow \mathcal{M}, v \models \varphi)$.

For any neighborhood frame (W, F), define its potential accessibility relation Acc_F:

$$w\, Acc_F\, v \quad \text{iff} \quad v \in \bigcap N_F(w)$$

Well-known:

Fact

(W,F) is Kripkean iff $\forall X \subseteq W\; F(X) = \{w \in W : (Acc_F)_w \subseteq X\}$

iff each $N_F(w)$ is a principal filter.

Carnap's Question for the meaning of $\Box$

Now take Kripkean (neighborhood) frames to be standard interpretations of $\Box$.

Carnap's Question for the meaning of $\Box$

Now take Kripkean (neighborhood) frames to be standard interpretations of $\Box$.

And ask what constraints force us to use Kripke semantics.

Carnap's Question for the meaning of $\Box$

Now take Kripkean (neighborhood) frames to be standard interpretations of $\Box$.

And ask what constraints force us to use Kripke semantics.

NB We are not trying to promote Kripke semantics as the 'right' semantics for modal logic.

Carnap's Question for the meaning of $\Box$

Now take Kripkean (neighborhood) frames to be standard interpretations of $\Box$.

And ask what constraints force us to use Kripke semantics.

NB We are not trying to promote Kripke semantics as the 'right' semantics for modal logic.

Rather, we note the centrality of Kripke semantics, and try to 'reverse engineer' its role by seeing how and when it originates from modal consequence relations.

A partial answer to Carnap's Question

K is the smallest normal modal logic.

A partial answer to Carnap's Question

K is the smallest normal modal logic.

Theorem

A finite local interpretation is Kripkean iff it is consistent with K*.*

A partial answer to Carnap's Question

K is the smallest normal modal logic.

Theorem

A finite local interpretation is Kripkean iff it is consistent with K.

This answers Carnap's Question for finite frames: K forces $\neg, \wedge, \Box$ to have their standard interpretation.

A partial answer to Carnap's Question

K is the smallest normal modal logic.

Theorem

A finite local interpretation is Kripkean iff it is consistent with K*.*

This answers Carnap's Question for finite frames: K forces $\neg, \wedge, \Box$ to have their standard interpretation.

$(Unif(Val^{fin}(\vdash^{\mathsf{K}})) = \{\neg, \wedge, \Box\})$

A partial answer to Carnap's Question

K is the smallest normal modal logic.

Theorem

A finite local interpretation is Kripkean iff it is consistent with K*.*

This answers Carnap's Question for finite frames: K forces $\neg, \wedge, \Box$ to have their standard interpretation.

$(Unif(Val^{fin}(\vdash^{\mathsf{K}})) = \{\neg, \wedge, \Box\})$

Unfortunately, this is very far from true for infinite domains.

A partial answer to Carnap's Question

K is the smallest normal modal logic.

Theorem

A finite local interpretation is Kripkean iff it is consistent with K.

This answers Carnap's Question for finite frames: K forces $\neg, \wedge, \Box$ to have their standard interpretation.

$(Unif(Val^{fin}(\vdash^{\mathsf{K}})) = \{\neg, \wedge, \Box\})$

Unfortunately, this is very far from true for infinite domains. The following is well-known.

Fact

The class of Kripkean local interpretations is not modally definable: there is no modal logic L *such that the neighborhood frames consistent with* L *are exactly the Kripkean ones.*

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad F_{\text{id}}(X) = X$$

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad F_{\text{id}}(X) = X$$

F_{uni} is 'metaphysical necessity': necessary truth = truth in all worlds.

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad F_{\text{id}}(X) = X$$

F_{uni} is 'metaphysical necessity': necessary truth = truth in all worlds. F_{id} is a trivial interpretation: necessary truth = truth.

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad\qquad F_{\text{id}}(X) = X$$

F_{uni} is 'metaphysical necessity': necessary truth = truth in all worlds. F_{id} is a trivial interpretation: necessary truth = truth.

Theorem

The only Kripkean permutation invariant interpretations over W are (W, F_{uni}), (W, F_{id}), and two more.

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad F_{\text{id}}(X) = X$$

F_{uni} is 'metaphysical necessity': necessary truth = truth in all worlds. F_{id} is a trivial interpretation: necessary truth = truth.

Theorem

The only Kripkean permutation invariant interpretations over W are (W, F_{uni}), (W, F_{id}), and two more.

This is too restricted: accessibility plays no role.

Permutation invariance is too strong

Definition

(W, F) is permutation invariant if $\pi(F) = F$ for all permutations π of W.

For example:

$$F_{\text{uni}}(X) = \begin{cases} W & \text{if } X = W \\ \emptyset & \text{otherwise} \end{cases} \qquad F_{\text{id}}(X) = X$$

F_{uni} is 'metaphysical necessity': necessary truth = truth in all worlds. F_{id} is a trivial interpretation: necessary truth = truth.

Theorem

The only Kripkean permutation invariant interpretations over W are (W, F_{uni}), (W, F_{id}), and two more.

This is too restricted: accessibility plays no role.

Exercise: Explain why permutation invariance is reasonable for first-order logic but not for modal logic.

Approach 1: stronger logics

In view of this, one approach is to go above K and look for (minimal) axioms that force interpretations to be Kripkean.

Approach 1: stronger logics

In view of this, one approach is to go above K and look for (minimal) axioms that force interpretations to be Kripkean.

Theorem (folklore, Wes Holliday p.c.)

If L *is a normal extension of* KB*, then all interpretations consistent with* L *are Kripkean.*

(The B axiom is $p \to \Box\Diamond p$, which on Kripke frames corresponds to symmetry, and in algebraic semantics to a residuation property.)

Approach 1: stronger logics

In view of this, one approach is to go above K and look for (minimal) axioms that force interpretations to be Kripkean.

Theorem (folklore, Wes Holliday p.c.)

If L *is a normal extension of* KB*, then all interpretations consistent with* L *are Kripkean.*

(The B axiom is $p \to \Box\Diamond p$, which on Kripke frames corresponds to symmetry, and in algebraic semantics to a residuation property.)

Theorem (Tadeusz Litak)

If L *is a normal extension of* $\mathrm{S4}$ *which is consistent only with Kripkean interpretations, then* $\mathrm{S5} \subseteq$ L*.*

Approach 1: stronger logics

In view of this, one approach is to go above K and look for (minimal) axioms that force interpretations to be Kripkean.

Theorem (folklore, Wes Holliday p.c.)

If L *is a normal extension of* KB, *then all interpretations consistent with* L *are Kripkean.*

(The B axiom is $p \to \Box\Diamond p$, which on Kripke frames corresponds to symmetry, and in algebraic semantics to a residuation property.)

Theorem (Tadeusz Litak)

If L *is a normal extension of* $\mathrm{S4}$ *which is consistent only with Kripkean interpretations, then* $\mathrm{S5} \subseteq$ L.

Open question: does this hold for K and KB, in place of S4 and S5?

Approach 2: explore truth locality

[T]opological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point. (van Benthem and Bezhanishvili, 2007).

Approach 2: explore truth locality

> *[T]opological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point*. (van Benthem and Bezhanishvili, 2007).

A topological frame is a neighborhood frame (W, int_τ), where τ is a topology on W and $int_\tau(X)$ is the interior of X.

Approach 2: explore truth locality

> *[T]opological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point*. (van Benthem and Bezhanishvili, 2007).

A topological frame is a neighborhood frame (W, int_τ), where τ is a topology on W and $int_\tau(X)$ is the interior of X. Equivalently, it is a neighborhood frame consistent with S4.

Approach 2: explore truth locality

> *[T]opological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point*. (van Benthem and Bezhanishvili, 2007).

A topological frame is a neighborhood frame (W, int_τ), where τ is a topology on W and $int_\tau(X)$ is the interior of X. Equivalently, it is a neighborhood frame consistent with S4.

The quote above means the following:

Fact

If $\mathcal{M}_\tau = (W, int_\tau, V)$ *and* $w \in A \in \tau$*, then* $\mathcal{M}_\tau, w \models \varphi \Leftrightarrow \mathcal{M}_{\tau_A}, w \models \varphi$.

Approach 2: explore truth locality

> *[T]opological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point*. (van Benthem and Bezhanishvili, 2007).

A topological frame is a neighborhood frame (W, int_τ), where τ is a topology on W and $int_\tau(X)$ is the interior of X. Equivalently, it is a neighborhood frame consistent with S4.

The quote above means the following:

Fact

If $\mathcal{M}_\tau = (W, int_\tau, V)$ *and* $w \in A \in \tau$*, then* $\mathcal{M}_\tau, w \models \varphi \Leftrightarrow \mathcal{M}_{\tau_A}, w \models \varphi$.

Here $\mathcal{M}_{\tau_A} = (A, int_{\tau_A}, V_A)$, where $\tau_A = \{X \cap A \colon X \in \tau\}$ is the subtopology induced by A, and $V_A(p) = V(p) \cap A$.

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

The van Benthem and Bezhanishvili observation amounts to invariance under generated submodels in topological semantics.

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

The van Benthem and Bezhanishvili observation amounts to invariance under generated submodels in topological semantics.

But this holds for all neighborhood models: for $A \subseteq W$, define

$$(A, G) \subseteq_g (W, F) \text{ iff } \forall X \subseteq W \; F(X) \cap A = G(X \cap A)$$

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

The van Benthem and Bezhanishvili observation amounts to invariance under generated submodels in topological semantics.

But this holds for all neighborhood models: for $A \subseteq W$, define

$$(A, G) \subseteq_g (W, F) \text{ iff } \forall X \subseteq W \; F(X) \cap A = G(X \cap A)$$

Fact

(a) *For Kripkean frames, $\subseteq_g$ is the usual notion.*

(b) *For topological frames, $(A, int_{\tau_A}) \subseteq_g (W, int_\tau)$ iff A is open.*

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

The van Benthem and Bezhanishvili observation amounts to invariance under generated submodels in topological semantics.

But this holds for all neighborhood models: for $A \subseteq W$, define

$$(A, G) \subseteq_g (W, F) \text{ iff } \forall X \subseteq W \; F(X) \cap A = G(X \cap A)$$

Fact

(a) *For Kripkean frames, $\subseteq_g$ is the usual notion.*

(b) *For topological frames, $(A, int_{\tau_A}) \subseteq_g (W, int_\tau)$ iff A is open.*

Proposition

If $(A, G) \subseteq_g (W, F)$, then for all V, $[\![\varphi]\!]_{(A,G,V_A)} = [\![\varphi]\!]_{(W,F,V)} \cap A$.

Invariance under generated submodels

In Kripke semantics truth is invariant under generated submodels.

The van Benthem and Bezhanishvili observation amounts to invariance under generated submodels in topological semantics.

But this holds for all neighborhood models: for $A \subseteq W$, define

$$(A, G) \subseteq_g (W, F) \text{ iff } \forall X \subseteq W \; F(X) \cap A = G(X \cap A)$$

Fact

(a) *For Kripkean frames, $\subseteq_g$ is the usual notion.*

(b) *For topological frames, $(A, int_{\tau_A}) \subseteq_g (W, int_\tau)$ iff A is open.*

Proposition

If $(A, G) \subseteq_g (W, F)$, then for all V, $[\![\varphi]\!]_{(A,G,V_A)} = [\![\varphi]\!]_{(W,F,V)} \cap A$.

This includes the above invariance facts as special cases.

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

If (W, F) is a neighborhood frame and $w \in W$, let

$$A^w = \bigcap \{X : (X, G) \subseteq_g (W, F) \text{ and } w \in X\}$$

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

If (W, F) is a neighborhood frame and $w \in W$, let

$$A^w = \bigcap\{X : (X, G) \subseteq_g (W, F) \text{ and } w \in X\}$$

Let $(W,F)[w] = (A^w, F_{A^w}) \quad (F_{A^w}(X) = F(X) \cap A^w \text{ for } X \subseteq A^w)$

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

If (W, F) is a neighborhood frame and $w \in W$, let

$$A^w = \bigcap\{X : (X, G) \subseteq_g (W, F) \text{ and } w \in X\}$$

Let $(W,F)[w] = (A^w, F_{A^w}) \quad (F_{A^w}(X) = F(X) \cap A^w \text{ for } X \subseteq A^w)$

Fact

If $(W, F)[w] \subseteq_g (W, F)$*, then* $(W, F)[w]$ *is the smallest gen. subframe containing* w *(= the usual point-gen. subframe if* (W, F) *is Kripkean).*

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

If (W, F) is a neighborhood frame and $w \in W$, let

$$A^w = \bigcap\{X : (X, G) \subseteq_g (W, F) \text{ and } w \in X\}$$

Let $(W,F)[w] = (A^w, F_{A^w})$ $\quad (F_{A^w}(X) = F(X) \cap A^w$ for $X \subseteq A^w)$

Fact

If $(W, F)[w] \subseteq_g (W, F)$, then $(W, F)[w]$ is the smallest gen. subframe containing w (= the usual point-gen. subframe if (W, F) is Kripkean).

Definition

(W, F) is strongly local if for every $w \in W$, $(W, F)[w] \subseteq_g (W, F)$.

Point-generated subframes

In Kripke semantics, but not in topological semantics, there is always a smallest generated subframe containing a given point.

If (W, F) is a neighborhood frame and $w \in W$, let

$$A^w = \bigcap \{X : (X, G) \subseteq_g (W, F) \text{ and } w \in X\}$$

Let $(W,F)[w] = (A^w, F_{A^w}) \quad (F_{A^w}(X) = F(X) \cap A^w \text{ for } X \subseteq A^w)$

Fact

If $(W, F)[w] \subseteq_g (W, F)$, then $(W, F)[w]$ is the smallest gen. subframe containing w (= the usual point-gen. subframe if (W, F) is Kripkean).

Definition

(W, F) is strongly local if for every $w \in W$, $(W, F)[w] \subseteq_g (W, F)$.

Theorem

All strongly local topological frames are Kripkean.

Bisimulation invariance

This result is still partial: there are strongly local non-Kripkean neighborhood frames consistent with K.

Bisimulation invariance

This result is still partial: there are strongly local non-Kripkean neighborhood frames consistent with K.

Another approach: presumably the most characteristic expression of locality in modal logic is in terms of bisimulations.

Bisimulation invariance

This result is still partial: there are strongly local non-Kripkean neighborhood frames consistent with K.

Another approach: presumably the most characteristic expression of locality in modal logic is in terms of bisimulations.

We can lift the Kripkean version of bisimulation to arbitrary neighborhood models, and show that a certain kind of invariance under this concept forces interpretations to be Kripkean.

K-bisimulations

Idea: regard two neighborhood models as similar from a Kripkean perspective if their associated Kripke models are bisimilar:

Definition

$Z \subseteq W \times W'$ is an k-bisimulation between pointed neighborhood models $\mathcal{M} = (W, F, V, w)$ and $\mathcal{M}' = (W', F', V', w')$ ('k' for 'Kripke'),

$$Z : \mathcal{M} \underline{\leftrightarrow}_k \mathcal{M}',$$

if Z is an ordinary bisimulation between (W, Acc_F, V, w) and $(W', Acc_{F'}, V', w')$.

Idea: regard two neighborhood models as similar from a Kripkean perspective if their associated Kripke models are bisimilar:

Definition

$Z \subseteq W \times W'$ is an k-bisimulation between pointed neighborhood models $\mathcal{M} = (W, F, V, w)$ and $\mathcal{M}' = (W', F', V', w')$ ('k' for 'Kripke'),

$$Z : \mathcal{M} \,\underline{\leftrightarrow}_k\, \mathcal{M}',$$

if Z is an ordinary bisimulation between (W, Acc_F, V, w) and $(W', Acc_{F'}, V', w')$. $\mathcal{M}$ and $\mathcal{M}'$ are k-bisimilar,

$$\mathcal{M} \,\underline{\leftrightarrow}_k\, \mathcal{M}'$$

if $Z : \mathcal{M} \,\underline{\leftrightarrow}_k\, \mathcal{M}'$ for some Z.

K-bisimulation invariance

Given (W, F, V), define the valuation $F(V)$ by $F(V)(p) = F(V(p))$.

K-bisimulation invariance

Given (W, F, V), define the valuation $F(V)$ by $F(V)(p) = F(V(p))$.

We say that an interpretation is bisimulation invariant if the action of F preserves bisimilarity:

Definition

(W, F) is k-bisimulation invariant if for all models (W, F, V, w) based on (W, F) and all other models (W', F', V', w'),

$$(W, F, V, w) \underline{\leftrightarrow}_k (W', F', V', w')$$

implies

$$(W, F, F(V), w) \underline{\leftrightarrow}_k (W', F', F'(V'), w')$$

K-bisimulation invariance

Given (W, F, V), define the valuation $F(V)$ by $F(V)(p) = F(V(p))$.

We say that an interpretation is bisimulation invariant if the action of F preserves bisimilarity:

Definition

(W, F) is k-bisimulation invariant if for all models (W, F, V, w) based on (W, F) and all other models (W', F', V', w'),

$$(W, F, V, w) \underline{\leftrightarrow}_k (W', F', V', w')$$

implies

$$(W, F, F(V), w) \underline{\leftrightarrow}_k (W', F', F'(V'), w')$$

Intuitively, k-bisimulation invariance says that Acc_F is the only thing that matters for the action of F.

K-bisimulation invariance

Given (W, F, V), define the valuation $F(V)$ by $F(V)(p) = F(V(p))$.

We say that an interpretation is bisimulation invariant if the action of F preserves bisimilarity:

Definition

(W, F) is k-bisimulation invariant if for all models (W, F, V, w) based on (W, F) and all other models (W', F', V', w'),

$$(W, F, V, w) \underline{\leftrightarrow}_k (W', F', V', w')$$

implies

$$(W, F, F(V), w) \underline{\leftrightarrow}_k (W', F', F'(V'), w')$$

Intuitively, k-bisimulation invariance says that Acc_F is the only thing that matters for the action of F. Indeed:

Theorem

A local interpretation is k-bisimulation invariant iff it is Kripkean.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic ‘extraction’ map.
- Semantic versions of ‘extraction’ promise to isolate logical constants and the logical part of a consequence relation.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.
- Semantic versions of 'extraction' promise to isolate logical constants and the logical part of a consequence relation.
- In particular, taking familiar relations of logical consequence as given, one might be able to justify the choice of logical constants in various familiar formal languages: Carnap's Question.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.
- Semantic versions of 'extraction' promise to isolate logical constants and the logical part of a consequence relation.
- In particular, taking familiar relations of logical consequence as given, one might be able to justify the choice of logical constants in various familiar formal languages: Carnap's Question.
- Clear and satisfactory results for first-order logic.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.
- Semantic versions of 'extraction' promise to isolate logical constants and the logical part of a consequence relation.
- In particular, taking familiar relations of logical consequence as given, one might be able to justify the choice of logical constants in various familiar formal languages: Carnap's Question.
- Clear and satisfactory results for first-order logic.
- New conceptual issues, and results and open problems, for modal logic.

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.
- Semantic versions of 'extraction' promise to isolate logical constants and the logical part of a consequence relation.
- In particular, taking familiar relations of logical consequence as given, one might be able to justify the choice of logical constants in various familiar formal languages: Carnap's Question.
- Clear and satisfactory results for first-order logic.
- New conceptual issues, and results and open problems, for modal logic.
- [Comparing these two is possibly interesting: role of principal filters; of permutation invariance; of symmetry (B axiom) vs. commutativity; notion of global quantifier/global interpretation.]

Summary

- (Sets of constants) and consequence relations are (Galois) dual concepts, via the Bolzano-Tarski map and a syntactic 'extraction' map.
- Semantic versions of 'extraction' promise to isolate logical constants and the logical part of a consequence relation.
- In particular, taking familiar relations of logical consequence as given, one might be able to justify the choice of logical constants in various familiar formal languages: Carnap's Question.
- Clear and satisfactory results for first-order logic.
- New conceptual issues, and results and open problems, for modal logic.
- [Comparing these two is possibly interesting: role of principal filters; of permutation invariance; of symmetry (B axiom) vs. commutativity; notion of global quantifier/global interpretation.]
- To do: other logics!

THANK YOU