Forcing as a tool to prove theorems

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LC2018 – Udine – 26/7/2018

We want to use *forcing* to prove ZFC-*derivable* theorems, rather than to establish ZFC-*independence* results.

A simple slogan which encompasses the basic ingredient for our proposal is:

If it is possible, it is true.

We will use *forcing* to show that certain mathematical statements are *possible*, and Woodin (actually the weaker Shoenfield) *absoluteness results* to argue that these statements must then be *true*.

Ultimately we will argue that Woodin's result and a variety of other set-theoretic principles are "natural" non-constructive principles which strengthen the axiom of choice.

Our approach to forcing outlines some of the consequences one can draw by accepting as true these non-constructive principles.

Outline of the talk

1. Two theorems proved using forcing
2. Forcing as a generalized version of Łoś theorem for ultraproducts
3. Generic absoluteness results as strong forms of Łoś theorem for ultrapowers
4. Generic absoluteness results for (fragments of) third order arithmetic

Section 1

Two theorems proved using forcing

Schanuel's conjecture

Let $F \subseteq K$ be fields.
For $a_1, \dots,_n \in K$

- $\mathrm{Td}_F(a_1, \dots, a_n) \geq k$ iff no k-sized subset of $\{a_1, \dots, a_n\}$ is a root of some polynomials with coefficients in F with k-variables.
- $\mathrm{Ldim}_F(a_1, \dots, a_n)$ is the linear dimension of the F-vector space spanned by $a_1, \dots, a_n$.

Conjecture (Schanuel)

For all $a_1, \dots, a_n \in \mathbb{C}$ $Td_{\mathbb{Q}}(a_1, \dots, a_n, e^{a_1}, \dots, e^{a_n}) \geq Ldim_{\mathbb{Q}}(a_1, \dots, a_n)$.

Schanuel's conjecture implies the algebraic transcendence of e and π since:

$$\mathrm{Td}_{\mathbb{Q}}(\pi, e) = \mathrm{Td}_{\mathbb{Q}}(i\pi, e) = \mathrm{Td}_{\mathbb{Q}}(1, i\pi, e, e^{i\pi}) \geq \mathrm{Ldim}_{\mathbb{Q}}(1, i\pi) = 2.$$

Definition

Let $F \subseteq K$ be fields and $E : (K, +) \to (K^*, \cdot)$ be a surjective homomorphism.

$$\mathrm{SC}(F, K) \equiv$$

$$\text{For all } a_{1,\ldots,n} \in K$$

$$\mathrm{Td}_F(a_1, \ldots, a_n, E(a_1), \ldots, E(a_n)) \geq \mathrm{Ldim}_F(a_1, \ldots, a_n).$$

Two results on Schanuel's conjecture

Zilber proved that Schanuel's conjecture is "consistent", more precisely: (K, E) is an exponential field if E is a surjective homomorphism of $(K, +)$ with $(K^*, \cdot)$.

Theorem (Zilber)

There is a theory T expressible in $L_{\omega_1,\omega}$ such that:

1. *Any model (K, E) of T is an algebraically closed exponential field;*
2. $\mathrm{SC}(\mathbb{Q}, K)$ *holds;*
3. *for any uncountable cardinal κ there is only one model of T in that cardinality.*

It is clearly open whether the K of size continuum is $\mathbb{C}$.

Two results on Schanuel's conjecture

Theorem (Kirby, Wilkie)

There is a countable field $F \subseteq \mathbb{C}$ such that $\mathrm{SC}(F, \mathbb{C})$ *holds.*

The result shows that Schanuel's conjecture holds modulo (eventually) a small error.

We can give an argument based on forcing for Kirby and Wilkie's result.

Using forcing and Shoenfield's absoluteness to prove Kirby and Wilkie's result

$\phi(K,\mathbb{C}) \equiv K \subseteq \mathbb{C}$ *is a countable field such that* $\mathrm{SC}(K,\mathbb{C})$ *holds.*

is a Π_1-statement over $(\mathbb{C},\mathbb{N},K,+,\cdot,x \mapsto e^x,R_1,\ldots,R_n)$ for suitable Borel predicates R_i, i.e. is a Π^1_1-statement.
A rough formulation of Shoenfield absoluteness results states

Whenever ψ is a Σ^1_2-statement, and ψ holds in some forcing extension of the universe of sets V, then ψ is true.

$\exists K\,\phi(K,\mathbb{C})$ is a Σ^1_2-statement, therefore it is enough to prove:

There exists a B *such that whenever* G *is* V*-generic for* B

1. $\mathrm{SC}(\mathbb{C}^V,\mathbb{C}^{V[G]})$ *holds.*
2. $[\![\mathbb{C}^V \text{is a countable field}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$.

If both conditions are fulfilled, $\exists K\,\phi(K,\mathbb{C})$ holds in $V[G]$, hence is true.

The second task is easily accomplished in set theory using the complete boolean algebra B given by the regular open subsets of the product space $\mathbb{C}^{\mathbb{N}}$ where $\mathbb{C}$ is endowed with the *discrete* topology (otherwise said the boolean completion of the forcing Coll$(\omega, 2^{\omega})$).
We are left with the first task.

Theorem (V.)

Assume B *is a complete boolean algebra and G is V-generic for* B*. Then* $\mathrm{SC}(\mathbb{C}^V, \mathbb{C}^{V[G]})$ *holds.*

Caveat

The above result is weaker than Kirby and Wilkie's original result, since it does not give an explicit description of K.
The proof uses heavily (as Wilkie and Kirby's proof do as well) a theorem of Ax establishing that $\mathrm{SC}(K, F)$ holds whenever K is the kernel of a derivation D on an exponential field F.

An outline of the proof would require us to dwelve more in depth into the correct formulation of Shoenfield's absoluteness, which is what we plan to do in the second part of the talk.

Borel incomparability of isomorphism and biembeddability for torsion abelian groups

Definition

Let (X,τ), (Y,σ) be Polish spaces (i.e. completely metrizable and separable) and $E \subseteq X^2, F \subseteq Y^2$.

$$E \leq_{\mathrm{Bor}} F$$

if there is a Borel function $f : X \to Y$ such that $x_0 \; E \; x_1$ if and only if $f(x_0) \; F \; f(x_1)$.

Assuming E, F are equivalence relations we have that

$$E \leq_{\mathrm{Bor}} F \text{ if and only if } |X/_E| = |Y/_F| \text{ is witnessed by a Borel function.}$$

This notion of reducibility is succesfully used to compare the complexity of classification problems. (Recall for example Tserunian's talk).

Let Tor $\subseteq 2^{(\mathbb{N}^3)}$ be the Borel subspace of the Cantor space given by countable torsion groups,i.e.

$G \subseteq 2^{(\mathbb{N}^3)}$ is in Tor if it is the graph of a binary operation $\cdot_G : \mathbb{N}^2 \to \mathbb{N}$ such that $(\mathbb{N}, \cdot_G, 0)$ is a torsion group (every element has finite order).

It can be checked that Tor with its inherited topology is a Polish space.

Define

- $\cong \subseteq \text{Tor}^2$ by $G \cong H$ if and only if G is isomorphic to H.
- $\equiv \subseteq \text{Tor}^2$ by $G \equiv H$ if and only if there are momonorphisms $f : G \to H$ and $k : H \to G$.

It is clear that $\cong$ and $\equiv$ are (analytic) equivalence relations.

Theorem (Caldironi, Thomas)

$$\cong \not\leq_{\text{Bor}} \equiv \not\leq_{\text{Bor}} \cong .$$

The statement

$$\phi \equiv \ (\cong \text{ and } \equiv \text{ are not Borel by-reducible})$$

is projective, i.e. Σ^1_n for some n (equivalently first-order definable in a structure of the form $(\mathbb{R}, P)$ with $P \subseteq \mathbb{R}^n$ a Borel relation).
We can use the following rough formulation of Woodin's generic absoluteness results:

Woodin

Assume there are class many Woodin cardinals.
Let ψ be a projective statement.
If ψ holds in some forcing extension it is true.

Using Woodin's absoluteness and large cardinals, if ϕ holds in some forcing extension it is true.
Caldironi and Thomas show that ϕ holds in some forcing extension.

With some work on the nature of the statement and the analysis of Caldironi and Thomas arguments one can drop the large cardinal assumptions (using ideas of Zapletal).

Section 2

Forcing as a generalized version of Łoś theorem for ultraproducts

Łoś theorem

Theorem

Let $\{\mathfrak{M}_l = (M_l, R_l) : l \in L\}$ *be first order models for* $\mathcal{L} = \{R\}$.
Let $G \subseteq \mathcal{P}(L)$ *be a ultrafilter on* L. *Set*

- $[f]_G = [h]_G$ *iff* $\{l \in L : f(l) = h(l))\} \in G$,
- $\bar{R}([f_1]_G, \ldots, [f_n]_G)$ *iff* $\{l \in L : R_l(f_1(l), \ldots, f_n(l))\} \in G$.

Then:

1. *For all* $\phi(x_1, \ldots, x_n)$ $(\prod_{l\in L} M_l/G, \bar{R}) \models \phi([f_1]_G, \ldots, [f_n]_G)$ *if and only if* $\{l \in L : \mathfrak{M}_l \models \phi(f_1(l), \ldots, f_n(l))\} \in G$.
2. *If* $\mathfrak{M}_l = \mathfrak{M}$ *for all* $l \in L$, $M \prec \prod_{l\in L} M_l/G$ *as witnessed by the map* $m \mapsto [c_m]_G$ *(where* $c_m : L \to M$ *is constant with value* m*).*

Recall on boolean algebras and Stone spaces

Given a boolean algebra B:

- St(B) is given by its ultrafilters G.
- St(B) is endowed with a *compact, Hausdorff* topology τ_{B} whose clopens are $N_b = \{G \in \mathrm{St}(\mathsf{B}) : b \in G\}$.
- The map $b \mapsto N_b$ defines a natural isomorphism of B with the boolean algebra CLOP(St(B)) of clopen subsets of St(B).

Given a topological space (X, τ):

- $A \subseteq X$ is *regular open* if $A = \text{Reg}(A)$, where $\text{Reg}(A) = \text{Int}(\text{Cl}(A))$ is the interior of the closure of A.
- $\text{RO}(X, \tau)$ is the family of *regular open* subsets of X and is always a complete boolean algebra with
  - $(A, B) \mapsto A \cap B$ giving the $\wedge$-operation on $\text{RO}(X)$,
  - $A \mapsto X \setminus \text{Cl}(A)$ giving the $\neg$-operation on $\text{RO}(X)$,
  - $\bigvee \{A_i : i \in I\} = \text{Reg}(\bigcup \{A_i : i \in I\})$ giving the supremum of any family of elements of $\text{RO}(X, \tau)$.

- B is *complete* if and only if

$$\text{CLOP}(\text{St}(\mathsf{B})) = \text{RO}(\text{St}(\mathsf{B}), \tau_{\mathsf{B}}) \cong \mathsf{B}.$$

- Spaces (X, τ) satisfying the property that its regular open sets are clopen are *extremally (or extremely) disconnected*.

- $\mathcal{P}(X)$ is a complete boolean algebra, and $\beta(X) = \text{St}(\mathcal{P}(X))$ is the Stone-Cech compactification of the space X endowed with discrete topology.
- $\beta(X)$ is compact and extremally disconnected and its algebra of clopen (regular open) subsets is exactly (isomorphic to) $\mathcal{P}(X)$.

Boolean valued models

Definition

Let B be a *cba* and a $\mathcal{L} = \left\{R_i : i \in I,\ c_j : j \in J\right\}$ be a first order *relational* language.
A B-*valued model* for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J\rangle$ with $c_j^{\mathcal{M}} \in M$ for all $j \in J$, and

$$\begin{aligned} =^{\mathcal{M}} : M^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![\tau = \sigma]\!], \end{aligned}$$

$$\begin{aligned} R^{\mathcal{M}} : M^n &\to \mathsf{B} \\ (\tau_1, \dots, \tau_n) &\mapsto [\![R(\tau_1, \dots, \tau_n)]\!]^{\mathcal{M}}_{\mathsf{B}} = [\![R(\tau_1, \dots, \tau_n)]\!] \end{aligned}$$

for $R \in \mathcal{L}$ an n-ary relation symbol.

Forcing relations on boolean valued models

The constraints on $R^{\mathcal{M}}$ and $=^{\mathcal{M}}$ are the following:

- for $\tau, \sigma, \chi \in M$,
  1. $[\![\tau = \tau]\!] = 1_{\mathsf{B}}$;
  2. $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!]$;
  3. $[\![\tau = \sigma]\!] \wedge [\![\sigma = \chi]\!] \leq [\![\tau = \chi]\!]$;
- for $R \in \mathcal{L}$ with arity n, and $(\tau_1, \ldots, \tau_n), (\sigma_1, \ldots, \sigma_n) \in M^n$,

$$[\![R(\tau_1, \ldots, \tau_n)]\!] \wedge \bigwedge_{h \in \{1,\ldots,n\}} [\![\tau_h = \sigma_h]\!] \leq [\![R(\sigma_1, \ldots, \sigma_n)]\!].$$

Boolean valued semantics

Definition

Let $\langle M, =^{\mathcal{M}}, R^{\mathcal{M}}\rangle$ be a B-valued model in the relational language $\mathcal{L} = \{R\}$, $\phi(x_1, \ldots, x_n)$ a $\mathcal{L}$-formula with displayed free variables, ν : free variables $\rightarrow M$.
$[\![\phi]\!]^{\mathcal{M},\nu}_{\mathsf{B}} = [\![\phi]\!]$, the *boolean value* of ϕ with the assignment ν is defined by recursion as follows:

- $[\![t = s]\!] = [\![\nu(t) = \nu(s)]\!]$,
  $[\![R(t_1, \ldots, t_n)]\!] = [\![R(\nu(t_1), \ldots, \nu(t_n))]\!]$;
- $[\![\neg\psi]\!] = \neg [\![\psi]\!]$;
- $[\![\psi \wedge \theta]\!] = [\![\psi]\!] \wedge [\![\theta]\!]$;
- $[\![\exists y\psi(y)]\!] = \bigvee_{\tau \in M} [\![\psi(y/\tau)]\!]$.

Soundness Theorem for B-valued semantics

Theorem (Soundness and Completeness Theorem)

Assume $\mathcal{L} = \left\{R_i : i \in I,\, c_j : j \in J\right\}$ *is a relational language,* T *an* $\mathcal{L}$*-theory, and* ϕ *an* $\mathcal{L}$*-formula.*

The following are equivalent:

- $T \vdash \phi$,
- *for any* B*-valued model* $\mathcal{M} = \langle M, R_i^M : i \in I,\, c_j^M : j \in J\rangle$ *and any valuation* $\nu : Var \to M$,

$$\bigwedge_{\psi \in T} [\![\psi]\!]_{\mathsf{B}}^{\mathcal{M},\nu} \leq [\![\phi]\!]_{\mathsf{B}}^{\mathcal{M},\nu}.$$

The completeness part of the theorem is automatic given that 2 is a complete boolean algebra.

Tarski quotient of B-valued models

Definition

Let B be a *cba*, $\mathcal{M}$ a B-valued model for $\mathcal{L}$, and G a ultrafilter over B. Consider the following equivalence relation on M:

$$\tau \equiv_G \sigma \Leftrightarrow [\![\tau = \sigma]\!] \in G$$

The first order (Tarski) model $\mathcal{M}/_G = \langle M/_G, R_i^{\mathcal{M}/_G} : i \in I, c_j^{\mathcal{M}/G} : j \in J\rangle$ is defined letting:

- For any n-ary relation symbol R in $\mathcal{L}$

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \ldots, [\tau_n]_G) \in (M/_G)^n : [\![R(\tau_1, \ldots, \tau_n)]\!] \in G\}.$$

- For any constant symbol c in $\mathcal{L}$

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

Full B-valued models

Definition

A B-valued model $\mathcal{M}$ for the language $\mathcal{L}$ is *full* if for every $\mathcal{L}$-formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$[\![\exists x \phi(x, \bar{\tau})]\!] = [\![\phi(\sigma, \bar{\tau})]\!]$$

Notice that in general $[\![\exists x \phi(x, \bar{\tau})]\!] = \bigvee_{\sigma \in M} \phi(\sigma, \bar{\tau})$ is only a supremum not a maximum! We want to avoid this unpleasant possibility.

Boolean valued Łoś Theorem — Forcing theorem

Theorem (B-valued Łoś's Theorem — Forcing theorem)

Assume $\mathcal{M}$ is a full B*-valued model for the relational language $\mathcal{L}$. Then for every formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$ and $(\tau_1, \ldots, \tau_n) \in M^n$:*

1. *For all ultrafilters G over* B*, $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ if and only if $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G$.*
2. *For all $a \in$ B the following are equivalent:*
    1. $[\![\phi(\tau_1, \ldots, \tau_n)]\!] \geq a$,
    2. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for all* $G \in N_a$,
    3. $\mathcal{M}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$ *for densely many* $G \in N_a$.

Łoś's Theorem versus boolean valued Łoś's Theorem

Fact

Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language $\mathcal{L}$. Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$-model, letting for each n-ary relation symbol $R \in \mathcal{L}$,
$[\![R(f_1, \ldots, f_n)]\!]_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x), \ldots, f_n(x))\}$.

Let G be any non-principal ultrafilter on X. Then the Tarski quotient N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G.

> *The usual Łoś theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$-valued model N of the boolean valued Łoś theorem.*
>
> *If N is an ultrapower of a model M, the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.*

Section 3

Generic absoluteness results as strong forms of Łoś theorem for ultrapowers

Boolean ultrapowers of compact Hausdorff spaces

Let X be a set with the discrete topology.

- For $a \in X$, $G_a \in \mathrm{St}(\mathcal{P}(X))$ is the principal ultrafilter of supersets of $\{a\}$ and $N_{\{a\}} = \{G_a\}$ is the clopen set of $\mathrm{St}(\mathcal{P}(X))$ given by an isolated point.
- The map $a \mapsto G_a$ embeds X as an open, dense, discrete subspace of $\mathrm{St}(\mathcal{P}(X))$.
- For any space (Y, τ), any $f : X \to Y$ is continuous. (Since X has the discrete topology)

Moreover if Y is compact Hausdorff:

- $f : X \to Y$ induces a unique continuous extension $\bar{f} : \mathrm{St}(\mathcal{P}(X)) \to Y$. ($\mathrm{St}(\mathcal{P}(X))$ is also the Stone-Cech compactification of X).
- $C(X, Y) = Y^X$ is canonically isomorphic to $C(\mathrm{St}(\mathcal{P}(X)), Y)$.
- $C(\mathrm{St}(\mathcal{P}(X)), Y) \cong Y^X$ can be endowed of the structure of a $\mathcal{P}(X)$-valued *elementary* extension of Y for any first order structure on Y.

What if we replace $\mathcal{P}(X)$ with an arbitrary (complete) boolean algebra?

Boolean ultrapowers of 2^ω

Let B be an arbitrary complete boolean algebra (so that $\mathsf{B} \cong \mathsf{RO}(\mathsf{St}(\mathsf{B}))$, and set

$$M = C(St(\mathsf{B}), 2^\omega).$$

Given $f_1, \ldots, f_n \in M$, let

$$f_1 \times \cdots \times f_n(G) = (f_1(G), \ldots, f_n(G)).$$

Fix R a Borel (Universally Baire) relation on $(2^\omega)^n$ (for example the equality relation Δ on $(2^\omega)^2$). Then its preimage

$$\{G : R(f_1(G) \ldots, f_n(G))\} = (f_1 \times \cdots \times f_n)^{-1}[R]$$

has the Baire property in St(B) (i.e. it has meager difference with a unique regular open set).

Define:

$$\begin{aligned} R^M : & M^n \to \mathsf{B} \\ & (f_1, \ldots, f_n) \mapsto \mathrm{Reg}\left(\{G : R(f_1(G), \ldots, f_n(G)\}\right). \end{aligned}$$

Boolean ultrapowers of 2^ω

Let B be an arbitrary (even atomless) complete boolean algebra. The following holds:

- For any Borel (universally Baire) relation R on $(2^\omega)^n$, the structure $(M, =^M, R^M)$ is a *full* B-valued model.
- For $G \in \mathrm{St}(\mathsf{B})$,

$$\begin{aligned} i_G : & 2^\omega \to M/G \\ & x \mapsto [c_x]_G \end{aligned}$$

  (c_x is the constant function with value x) defines an injective morphism $(2^\omega, R)$ into $(M/G, R^M/G)$.

It is not clear whether this morphism is an elementary map or not:

- This is the case for $\mathsf{B} = \mathcal{P}(X)$, since in this case we are analyzing the standard embedding of the first order structure $(2^\omega, R)$ in its ultrapowers induced by ultrafilters on $\mathcal{P}(X)$.
- What are the properties of this map if B is some other complete (atomless) boolean algebra?

Shoenfield's absoluteness rephrased

Theorem (Cohen and Woodin's absoluteness)

Assume B *is a complete boolean algebra and* $R \subseteq (2^\omega)^n$ *is a Borel (Universally Baire) relation. Let* $M = C(St(\mathsf{B}), 2^\omega)$ *and* $G \in \mathrm{St}(\mathsf{B})$. *Then*

$$(2^\omega, =, R) \prec_{\Sigma_2} (M/_G, =^M /_G, R^M/_G).$$

If one assumes the existence of class many Woodin cardinals

$$(2^\omega, =, R) \prec (M/_G, =^M /_G, R^M/_G).$$

Proof.

$C(St(\mathsf{B}), 2^\omega)$ is isomorphic to the B-names in V^{B} for elements of 2^ω (see next slide). Apply Shoenfield's (or Woodin's) absoluteness to V and $V[H]$ (for H V-generic for B) to infer the desired conclusion. □

$C(\mathrm{St}(\mathsf{B}), 2^\omega)$ and V^{B}

Given $f \in C(\mathrm{St}(\mathsf{B}), 2^\omega) = M$, $\sigma \in V^{\mathsf{B}}$ with $[\![\sigma \in 2^\omega]\!] = 1_{\mathsf{B}}$ define:

- $\tau_f = \left\{\langle\langle n, i\rangle, f^{-1}[N_{n,i}]\rangle : n < \omega, i < 2\right\} \in V^{\mathsf{B}}$,
- $g_\sigma \in M$ by $g_\sigma(G)(n) = i$ iff $[\![\sigma(n) = i]\!] \in G$.

Then

- $g_{\tau_f} = f$,
- $\left[\!\left[\tau_{g_\sigma} = \sigma\right]\!\right] = 1_{\mathsf{B}}$.

These identities allow to translate forcing relations from both sides.

The lift of a Universally Baire relation R to V^{B} is translated as the forcing relation (on M)

$$\begin{aligned} R^M : & M^n \to \mathsf{B} \\ & (f_1, \ldots, f_n) \mapsto \mathrm{Reg}\left(\{G : R(f_1(G), \ldots, f_n(G)\}\right). \end{aligned}$$

Universal Baireness grants that the lift R^M behaves as desired.

Section 4

Generic absoluteness results for (fragments of) third order arithmetic

Looking at 2^ω is the same as looking at H_{ω_1}

There exists a natural correspondence between the theory of projective subsets of 2^ω and the first order theory of H_{ω_1}. Any Σ^1_{n+1}-property of 2^ω corresponds to a Σ_n-definable property for the structure $(H_{\omega_1}, \in, =)$.
Moreover 2^ω is a definable class in H_{ω_1}, hence the first order theory of H_{ω_1} interprets that of 2^ω with projective predicates.
The converse holds as well.
Hence it is essentially the same to look at the first order theory of 2^ω or at the first order theory of H_{ω_1}.

Boolean ultrapowers of H_{ω_2}

To analyze how to use forcing for the analysis of compact spaces other than 2^ω it is more convenient to move from H_{ω_1} to the analysis of H_κ for larger κ.

If we can define *elementary* boolean ultrapowers of H_κ, we can naturally define *elementary* boolean ultrapowers of any compact Hausdorff Y (or more generally any mathematical structure) definable in H_κ.

Let us now mention a few generic absoluteness results for the first order theory of H_{ω_2}.
This structure captures most of the problems formalizable in third order arithmetic, among which the continuum problem.

Remark that:

- CH holds iff H_{ω_2} models CH.
- $\neg$CH is not first order expressible in the first order theory of the structure $(H_{\omega_1}, \in)$.

Stationary set preserving forcings and forcing axioms

Definition

$S \subseteq \omega_1$ is *stationary* if for every $f : \omega_1 \to \omega_1$ there is $\alpha \in S$ such that $f[\alpha] \subseteq \alpha$.

Definition

B is *stationary set preserving* (SSP) if and only if whenever G is V-generic for B and $S \subseteq \omega_1$ is stationary in V, $V[G]$ models that S is stationary.

(It is harder for S to remain stationary in $V[G]$ since there are more functions $f : \omega_1 \to \omega_1$ for which a fixed point $\alpha \in S$ must be found.)

Definition

For a topological space X, $\mathrm{FA}_{\omega_1}(X)$ holds if for all families $\{D_\alpha : \alpha < \omega_1\}$ of dense open subsets of X

$$\bigcap_{\alpha<\omega_1} D_\alpha \neq \emptyset.$$

(the above is a strong form of the Baire category theorem for X.)

Stationary set preserving forcings

The following holds:

- There are non-stationary set preserving forcings (for example $\mathrm{Coll}(\omega, \omega_1)$).
- If $\mathsf{B} \notin \mathsf{SSP}$, $\mathsf{FA}_{\omega_1}(\mathsf{St}(\mathsf{B}))$ fails.
- Martin's Maximum MM states that:

  $\mathsf{FA}_{\omega_1}(\mathsf{St}(\mathsf{B}))$ *holds if and only if* $\mathsf{B} \in \mathsf{SSP}$*.*

- MM is consistent relative to the existence of a supercompact cardinal.
- MM solves many interesting problems formalizable in third order arithmetic (Whitehead problem for groups, Suslin hypothesis, CH, the existence of outer automorphisms for the Calkin algebra, etc....).
- MM entails that whenever G is V-generic for some B

$$(H^V_{\omega_1}, \in) \prec_{\Sigma_1} (H^{V[G]}_{\omega_1}, \in) \text{ if and only if } \mathsf{B} \in \mathsf{SSP}.$$

Strong forms of MM yield generic absoluteness

I’ve a long list of generic absoluteness results for third order arithmetic assuming strong forms of Martin’s maximum (some of them by myself, some of them with Giorgio Audrito -a former PhD student, some of them with David Asperò).
I will just give one example.

Strong forms of MM yield generic absoluteness

Theorem (V.)

There is an axiom MM^{+++} *such that:*

- MM^{+++} *is consistent relative to the existence of a super 2-huge cardinal.*
- MM^{+++} *entails* MM*.*
- *Whenever* $\mathsf{B} \in \mathsf{SSP}$ *preserves* MM^{+++} *and G is V-generic for* B

$$H_{\omega_2} \prec H_{\omega_2}^{V[G]}.$$

- *Assume* MM^{+++}*. Then there are saturated models of* $Th(H_{\omega_2})$ *of arbitrarily large size of the form* $H_{\omega_2}^{\mathsf{B}}/_G$ *for some* $G \in \mathrm{St}(\mathsf{B})$ *and* $\mathsf{B} \in \mathsf{SSP}$*.*
  ($H_{\omega_2}^{\mathsf{B}} = \left\{\tau \in V^{\mathsf{B}} : [\![\tau \in H_{\dot{\omega}_2}]\!] = 1_{\mathsf{B}}\right\}$*.)*

Section 5

References

Surveys (less or more detailed) on (some parts of) this presentation:

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