Complex Exponential Field

Paola D'Aquino

Università degli Studi della Campania "L. Vanvitelli"

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Exponential Function

The model theoretic analysis of the exponential function over a field started with a problem left open by Tarski in the 30's, about the decidability of the reals with exponentiation. Only in the mid 90's Macintyre and Wilkie gave a positive answer to this question assuming Schanuel's Conjecture. Complex exponentiation involves much deeper issues, and it is much harder to approach, as it inherits the Gödel incompleteness and undecidability phenomena via the definition of the set of periods. Despite this negative results there are still many interesting and natural model-theoretic aspects to analyze.

Exponential rings

Definition: An exponential ring, or E-ring, is a pair (R, E) where R is a ring (commutative with 1) and

$$E : (R, +) \to (\mathcal{U}(R), \cdot)$$

a morphism of the additive group of R into the multiplicative group of units of R satisfying

- $E(x + y) = E(x) \cdot E(y)$ for all $x, y \in R$
- $E(0) = 1$.

1. $(\mathbb{R}, exp)$; $(\mathbb{C}, exp)$;
2. (K, E) where K is any ring and $E(x) = 1$ for all $x \in K$.

Model-theoretic analysis of $(\mathbb{C}, +, \cdot, 0, 1,)$

The model theory of the field of complex numbers is very *tame*

$\mathbb{C}$ is a canonical model of $ACF(0)$, i.e. it is the unique algebraically closed field of characteristic 0 and cardinality $2^{\aleph_0}$

$\mathbb{C}$ is *strongly minimal,* i.e. the definable subsets of $\mathbb{C}$ are either finite or cofinite.

The definable sets have a geometrical interpretation in terms of algebraic varieties.

Comparing $(\mathbb{R}, exp)$ and $(\mathbb{C}, exp)$

$Th(\mathbb{R}, exp)$ decidable
modulo (SC)
(Macintyre-Wilkie '96)

$Th(\mathbb{C}, exp)$ undecidable
$\mathbb{Z} = \{x : \forall y(E(y) = 1 \to E(xy) = 1)\}$

$Th(\mathbb{R}, exp)$
model-complete
(Wilkie '96)

$Th(\mathbb{C}, exp)$
not model-complete
(Macintyre, Marker)

$Th(\mathbb{R}, exp)$ o-minimal
good description of
definable sets
(Wilkie '96)

- Is $Th(\mathbb{C}, exp)$ quasi-minimal?
- What are the automorphisms of $(\mathbb{C}, exp)$?
- Is $\mathbb{R}$ definable in $(\mathbb{C}, exp)$?

$(\mathbb{C}, exp)$

Macintyre 1996

$x \in \mathbb{Q}$ iff $\exists t, u, v((v-u)t = 1 \wedge E(v) = E(u) = 1 \wedge (vx = u))$

Laczkovich 2002

For any $x \in \mathbb{Q}$

$x \in \mathbb{Z}$ iff $\exists z(E(z) = 2 \wedge E(zx) \in \mathbb{Q})$

$\mathrm{Th}_{\exists}(\mathbb{C}, exp)$ is undecidable.

Model-theoretic analysis of $(\mathbb{C}, exp)$

What can we say about $(\mathbb{C}, exp)$?

Zilber Conjecture: $(\mathbb{C}, exp)$ is quasi-minimal, i.e. every subset of $\mathbb{C}$ definable in $(\mathbb{C}, exp)$ is either countable or co-countable.

In 2004 Zilber introduces a new class of E-fields, the *pseudo-exponential fields* (or *Zilber fields*), and finds an axiomatization of this class in $\mathcal{L}_{\omega_1\omega}(Q)$

- Q is the quantifier *exist uncountably many*

- $L_{\omega_1\omega}$ allows countable $\vee$ and $\wedge$

Axiomatization of Zilber

(K, E) is a **pseudo-exponential fields (or Zilber fields)** if:

- K is an algebraically closed field of characteristic 0;
- $E : (K, +, 0) \longrightarrow (K^{\times}, \cdot, 1)$ is a surjective homomorphism and there is $\omega \in K$ transcendental over $\mathbb{Q}$ such that $\ker E = \mathbb{Z}\omega$;
- **Schanuel's Conjecture (SC)** Let $\lambda_1, \ldots, \lambda_n \in K$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, E(\lambda_1), \ldots, E(\lambda_n))$ has transcendence degree (t.d.) at least n over $\mathbb{Q}$;

Axiomatization of Zilber

- **(Strong Exponential Closure)** For all finite $A \subseteq K$ if $V \subseteq G_n(K) = K^n \times (K^*)^n$ is irreducible, free and normal with $\dim V = n$ there is $(\overline{z}, E(\overline{z})) \in V$ generic over A.
- **(Countable Closure Property)** For all finite $A \subseteq K$, the exponential algebraic closure $ecl^K(A)$ of A in K is countable

$V \subseteq G_n(K)$ is normal if $dim[M]V \geq k$ for any $k \times n$ integer matrix M of rank k.

$V \subseteq G_n(K)$ is free if there are no $m_1, \ldots, m_n \in \mathbb{Z}$ and $a, b \in K$ where $b \neq 0$ such that V is contained in $\{(\overline{x}, \overline{y}) : m_1 x_1 + \ldots + m_n x_n = a\}$ or $\{(\overline{x}, \overline{y}) : y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} = b\}$.

Pseudo-exponential fields - Categoricity

Theorem (Zilber 2000/2005 - Bays and Kirby 2013/2018)

Up to isomorphism, there is exactly one model of the axioms in each uncountable cardinality.

Theorem (Zilber, Bays and Kirby)

- Each pseudo exponential field is quasiminimal, i.e. every definable set is countable or co-countable.
- If (K, E) is pseudo-exponential field, $|K| = k$ then there are 2^k many automorphisms of (K, E).

Zilber's Conjecture: The unique pseudo-exponential field of cardinality $2^{\aleph_0}$ is $(\mathbb{C}, exp)$.

A positive answer would imply

• Is $(\mathbb{C}, exp)$ quasi-minimal? YES

• Are there automorphisms of $(\mathbb{C}, exp)$ different from identity and conjugation? YES

• Is $\mathbb{R}$ definable in $(\mathbb{C}, exp)$ NO

$(\mathbb{C}, exp)$ vs $(\mathbb{C}, E)$

- Does $(\mathbb{C}, exp)$ satisfy properties which follow directly from Zilber's axioms?

- Does a pseudo-exponential field (K, E) satisfy properties which are known for $(\mathbb{C}, exp)$?

- Analytic method and results cannot be applied over pseudo-exponential fields, no topology except an obvious exponential Zariski topology

Generalization of Lindemann-Weierstrass Theorem:
Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers which are linearly independent over $\mathbb{Q}$.Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraic independent over $\mathbb{Q}$.

1. $\alpha = 1$ transcendence of e (Hermite 1873)
2. $\alpha = 2\pi i$ transcendence of π (Lindemann 1882)
3. $\overline{\alpha} = (\pi, i\pi)$ then tr.d.$(\pi, i\pi, e, e^{i\pi}) = 2$, i.e. π, e^{π} are algebraically independent over $\mathbb{Q}$ (Nesterenko 1996)
4. (SC) is true for power series $\mathbb{C}[[t]]$ (Ax 1971)

$(\mathbb{C}, exp)$ vs $(\mathbb{C}, E)$

Theorem (Zilber 2005)

$(\mathbb{C}, exp)$ satisfies the countable closure property.

Remark

Assuming Schanuel's Conjecture the axiom of Strong Exponential Closure for $(\mathbb{C}, exp)$ is the only impediment to prove Zilber's Conjecture.

Theorem (Mantova 2011)

If (K, E) is a pseudo-exponential field of cardinality up to the continuum then there exists an involution σ on K, i.e. there is a field automorphism $\sigma : K \longrightarrow K$ of order 2 such that $\sigma \circ E = E \circ \sigma$

Exponential polynomials over $\mathbb{C}$ with finitely many roots

Theorem (Henson and Rubel 1984)

Let $f(\overline{X}) \in \mathbb{C}[\overline{X}]^E$.

$$f(\overline{X}) \text{ has no solution in } \mathbb{C} \text{ iff } f(\overline{X}) = e^{g(\overline{X})}$$

where $g(\overline{X}) \in \mathbb{C}[\overline{X}]^E$.

Theorem (Katzberg 1983)

A non constant exponential polynomial $f(z) \in \mathbb{C}[z]^E$ has always infinitely many zeros unless it is of the form

$$f(z) = (z - \alpha_1)^{n_1} \cdot \ldots \cdot (z - \alpha_k)^{n_k} e^{g(z)},$$

where $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, $n_1, \ldots, n_k \in \mathbb{N}$, and $g(z) \in \mathbb{C}[z]^E$.

Theorem (D'A., Macintyre and Terzo, 2010)

Let (K, E) be a Zilber field and $f(\overline{X}) \in K[\overline{X}]^E$.

$$f(\overline{X}) \text{ has no solution in } K \text{ iff } f(\overline{X}) = e^{g(\overline{X})}$$

where $g(\overline{X}) \in K[\overline{X}]^E$.

Theorem (D'A., Macintyre and Terzo, 2010)

A non constant exponential polynomial $f(z) \in K[z]^E$ has always infinitely many zeros unless it is of the form

$$f(z) = (z - \alpha_1)^{n_1} \cdot \ldots \cdot (z - \alpha_k)^{n_k} e^{g(z)},$$

where $\alpha_1, \ldots, \alpha_k \in K$, $n_1, \ldots, n_k \in \mathbb{N}$, and $g(z) \in K[z]^E$.

$(\mathbb{C}, exp)$ vs (K, E)

Corollary (Picard's Little Theorem)

Let $f(x) \in K[x]^E$. If $f(x)$ is non constant then $f(x)$ cannot omit two values.

Generic solutions

Notation. Let $e_0(z) = z$, and for every $k \in \mathbb{N}$, $e_{k+1}(z) = e^{e_k(z)}$.
Fix $1 \leq k \in \mathbb{N}$, let $\overline{x} = (x_0, \ldots, x_k)$ and $p(\overline{x}) \in \mathbb{C}[\overline{x}]$.
Let $f(z) = p(z, e_1(z), \ldots, e_k(z))$ over $\mathbb{C}$.

Definition

A solution a of $f(z) = 0$ is **generic** over L (for L a finitely generated extension of $\mathbb{Q}$ containing the coefficients of p) if

$$t.d._L(a, e_1(a), \ldots, e_k(a)) = k,$$

where k is the number of iterations of exponentiation which appear in the polynomial p.

Strong exponential closure - simplest case

THEOREM (Marker 2006)

1) If $p(x, y) \in \mathbb{C}[x, y]$ is irreducible and depends on x and y then $f(z) = p(z, e^z)$ has infinitely many zeros.

2) **(SC)** If $p(x, y) \in \mathbb{Q}^{alg}[x, y]$ then there are infinitely many algebraically independent solutions over $\mathbb{Q}$.

Proof

1) Existence of infinitely many zeros follows from Hadamard Factorization theorem and Henson and Rubel's result.

2) Schanuel's Conjecture is crucial.

Strong exponential closure - simplest case

Theorem (Mantova 2016)

(SC) If $p(x, y) \in \mathbb{C}[x, y]$ is irreducible and depends on x and y then there is $z \in \mathbb{C}$ such that $p(z, e^z) = 0$ and $tr.d.(z, e^z/K) = 1$ for any finitely generated $K \subset \mathbb{C}$.

Proof uses some number theory results due to Zannier in order to show that there are only finitely many solutions of p in K^{alg}.

Ideas due also to Gunaydin and Martin-Pizarro.

Iterated exponentials

Question: Let $p(x, y_1, \ldots, y_n) \in \mathbb{Q}^{alg}[x, y_1, \ldots, y_n]$ a nonzero irreducible polynomial depending on x and the last variable y_n. Does

$$p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}})) = 0$$

have a generic solution?

Remark

Strong Exponential Closure in $\mathbb{C}$ would imply a positive answer.

Iterated exponentials

Theorem (D'A., Fornasiero and Terzo, 2017)

(SC) Let $p(x, y_1, y_2, y_3) \in \mathbb{Q}^{alg}[x, y_1, y_2, y_3]$ be a nonzero irreducible polynomial depending on x and y_3. Then, there exists a generic solution of

$$p(z, e^z, e^{e^z}, e^{e^{e^z}}) = 0.$$

Due to Katzberg's result the polynomial has infinitely many solutions, unless it is of a certain form. No restrictions on the coefficients and no (SC).

In the proof of the theorem we use a refinement of a result due to Masser on the existence of zeros of systems of exponential equations

Masser's result

Theorem

Let $P_1(\overline{x}), \dots, P_n(\overline{x}) \in \mathbb{C}[\overline{x}]$, where $\overline{x} = (x_1, \dots, x_n)$, and $P_i(\overline{x})$ are non zero polynomials in $\mathbb{C}[\overline{x}]$. Then there exist $z_1, \dots, z_n \in \mathbb{C}$ such that

$$\begin{cases} e^{z_1} = P_1(z_1, \dots, z_n) \\ e^{z_2} = P_2(z_1, \dots, z_n) \\ \vdots \\ e^{z_n} = P_n(z_1, \dots, z_n) \end{cases} \tag{1}$$

We have to show that the function $F : \mathbb{C}^n \to \mathbb{C}^n$

$F(x_1, \dots, x_n) = (e^{x_1} - P_1(x_1, \dots, x_n), \dots, e^{x_n} - P_n(x_1, \dots, x_n))$

has a zero in $\mathbb{C}^n$.

Lemma (Kantorovich)

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an entire function, and $\overline{p}_0$ be such that $J(\overline{p}_0)$, the Jacobian of F at $\overline{p}_0$, is non singular. Let $\eta = |J(\overline{p}_0)^{-1}F(\overline{p}_0)|$ and U the closed ball of center p_0 and radius 2η. Let $M > 0$ be such that $|H(F)|^2 \leq M^2$ (where $H(F)$ denotes the Hessian of F). If $2M\eta|J(\overline{p}_0)^{-1}| < 1$ then there is a zero of F in U.

Lemma

Let $P_1(\overline{x}), \ldots, P_n(\overline{x}) \in \mathbb{C}[\overline{x}]$, where $\overline{x} = (x_1, \ldots, x_n)$ and $d_1, \ldots, d_n$ be the total degrees of $P_1(\overline{x}), \ldots, P_n(\overline{x})$, respectively. There exists a constant $c > 0$ and an infinite set $S \subseteq \mathbb{Z}^n$ such that

$$|P_j(2\pi i k_1, \ldots, 2\pi i k_n)| \geq c(1 + \sum_{l=1}^{n} |k_l|)^{d_j}$$

for all $\overline{k} = (k_1, \ldots, k_n) \in S$, $j = 1, \ldots, n$.

First example

Let $f(z) = p(z, e^{e^z})$ where $0 \neq p(x, y) \in \mathbb{C}[x, y]$, and both x and y appear in p. We want to show that it has a generic solution in $\mathbb{C}$.

For this purpose we consider the corresponding system in four variables (z_1, z_2, w_1, w_2):

$$V = \begin{cases} p(z_1, w_2) = 0 \\ w_1 = z_2 \end{cases} \tag{2}$$

thought of as an algebraic set V in $G_2(\mathbb{C}) = \mathbb{C}^2 \times (\mathbb{C}^*)^2$.

First example

Theorem

(SC) If $p(x, y) \in \mathbb{Q}^{alg}[x, y]$ then the variety V intersects the graph of exponentation in a generic point (w, e^w, e^w, e^{e^w}), (i.e. $t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) = 2 = dimV$).

Proof: Wlog let w be a non-zero solution.
Case 1: w and e^w are linearly independent. Schanuel's Conjecture implies:

$$t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) \geq 2.$$

Indeed, the transcendence degree is exactly 2 since w and e^{e^w} are algebraically dependent. Hence, $(w, e^w, e^w, e^{e^w}) \in V$ and $t.d._{\mathbb{Q}}(w, e^w, e^w, e^{e^w}) = 2$, which is the dimension of V, and so the point (w, e^w, e^w, e^{e^w}) is generic for V.

Case 2. w, e^w are linearly dependent over $\mathbb{Q}$, i.e.

$$ne^w = mw \tag{3}$$

for some $m, n \in \mathbb{Z}$ and $(m, n) = 1$, and $w \neq 0$ implies $n \neq 0$. Moreover, w is transcendental over $\mathbb{Q}$, otherwise we have a contradiction with Lindemann Weierstrass Theorem. Applying exponentation to relation (3) it follows

$$e^{ne^w} = e^{mw},$$

i.e.

$$(e^{e^w})^n = (e^w)^m = (\frac{m}{n}w)^m = (\frac{m}{n})^m w^m.$$

Subcase 2.1: If $n, m > 0$ then (w, e^{e^w}) is a root of $q(x, y) = sx^m - y^n$ where $s \in \mathbb{Q}$

Subcase 2.2: If $n > 0$ and $m < 0$ then (w, e^{e^w}) is a root of $q(x, y) = x^{-m}y^n - r$, where $r \in \mathbb{Q}$

In both cases $q(x, y)$ is irreducible, and it can be showed that p divides q. Since p and q are irreducible polynomials they differ by a non-zero constant.
So, wlog $p(x, y) = sx^m - y^n$ (or $p(x, y) = x^{-m}y^n - r$).

For any solution (w, e^{e^w}) of $p(x, y) = 0$ the linear dependence between w and e^w is uniquely determined by the degrees of x and y in p, hence s is uniquely determined.

Using a system a' la Masser it is always possible to find a point (w, e^w, e^w, e^{e^w}) of V with w, e^w linearly independent. Indeed, we want $p(z, e^{e^z}) = 0$ and $z \neq se^z$, $s \in \mathbb{Q}$.

It is enough to solve the following system a la Masser

$$\begin{cases} e^z = A(z,t,u) \\ e^u = B(z,t,u) \\ e^t = C(z,t,u) \end{cases} \tag{4}$$

where $A(z,t,u) = \frac{t}{m}$, $B(z,t,u) = \frac{t}{m} - sz$, and $C(z,t,u) = \frac{z^n}{s}$.

By Masser's result there exists a solution of system (4) which gives a generic solution of the original polynomial, generic thanks to the second equation in (4) which guarantees that there is no linear dependence between a solution z and its exponential e^z.

Generalization of Masser's result

A cone is an open subset $U \subseteq \mathbb{C}^n$ s. t. for every $1 \leq t \in \mathbb{R}$, if $\overline{x} \in U$ then $t\overline{x} \in U$.

Definition

An algebraic function is an analytic function $f : U \to \mathbb{C}$ s. t. there exists a nonzero polynomial $q(\overline{x}, u) \in \mathbb{C}[\overline{x}, u]$ with $q(\overline{x}, f(\overline{x})) = 0$ on all $\overline{x} \in U$. If, moreover, the polynomial q is monic in u, we say that f is integral algebraic.

Theorem

Let $f_1, \ldots, f_n : U \to \mathbb{C}$ be nonzero algebraic functions, defined on some cone U. Assume that $U \cap (2\pi i \mathbb{Z}^*)^n$ is Zariski dense in $\mathbb{C}^n$. Then

$$\begin{cases} e^{z_1} = f_1(\overline{z}), \\ \ldots \\ e^{z_n} = f_n(\overline{z}) \end{cases} \tag{5}$$

has a solution $\overline{a} \in U$.

Second example

Let $f(z) = p(z, e^z, e^{e^z}) \in \mathbb{C}[x, y]$, and consider the corresponding system in four variables (z_1, z_2, w_1, w_2):

$$V = \begin{cases} p(z_1, z_2, w_2) = 0 \\ w_1 = z_2 \end{cases} \qquad (6)$$

thought of as an algebraic set V in $G_2(\mathbb{C})$.

Theorem

(SC) If $p(x, y, z) \in \mathbb{Q}^{alg}[x, y, z]$, then the variety V defined in (6) intersects the graph of exponentation in a generic point.

Proof: Let $a \neq 0$ be a root of f. If a and e^a are linearly independent then as before by (SC) $t.d._{\mathbb{Q}}(a, e^a, e^a, e^{e^a}) = 2$, so that (a, e^a, e^a, e^{e^a}) is a generic point of V.

Second example

If $e^a = ra$ for some $r \in \mathbb{Q}$ then a is necessarily transcendental by Lindemann-Weierstrass.

Call $r \in \mathbb{Q}$ "bad" if there exists $a \in \mathbb{C}$ solution of (6), s. t. $e^a = ra$.

Using an argument similar to that of the previous case we can prove that there exist only finitely many "bad" $r \in \mathbb{Q}$, $\{r_1, \ldots, r_k\}$.

We construct a generalization of the system a la Masser whose solution guarantees the existence of a point in V whose t.d. is 2.

Second example

$$\begin{cases} e^z = f_1(z, t, u_1, \dots, u_k) \\ e^t = f_2(z, t, u_1, \dots, u_k) \\ e^{u_1} = f_3(z, t, u_1, \dots, u_k) \\ \dots \\ e^{u_k} = f_{k+2}(z, t, u_1, \dots, u_k) \end{cases}$$

where $f_1 = t, f_3 = t - r_1 z, \dots f_{k+2} = t - r_k z$, and f_2 is the algebraic function which solves z in the original polynomial $p(x, y, z) = 0$. By Masser's generalization the above system has a solution (b, e^b, e^b, e^{e^b}) which turns out to be a generic solution for (6), since the last k equations guarantee that there is no linear dependence between b and e^b.

Three iterations

In an analogous way we can prove

Let $f(z) = p(z, e^z, e^{e^z}, e^{e^{e^z}})$. The corresponding system in six variables $(z_1, z_2, z_3, w_1, w_2, w_3)$ is:

$$V = \begin{cases} p(z_1, z_2, z_3, w_3) = 0 \\ w_1 = z_2 \\ w_2 = z_3. \end{cases} \tag{7}$$

thought of as an algebraic set V in $G_2(\mathbb{C})$.

Theorem (DFT '17)

(SC) If $p(x, y, z, w) \in \mathbb{Q}^{alg}[x, y, z, w]$ then the variety V has a generic point.

Strong exponential closure

We now consider the more general case of a variety and not a single polynomial.

Theorem (DFT '18)

(SC) Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ an irreducible variety over $\mathbb{Q}^{alg}$ with $\dim V = n$. If $\pi_1(V)$ and $\pi_2(V)$ are dominant, then there exists $\overline{a} \in \mathbb{C}^n$ such that $(\overline{a}, e^{\overline{a}}) \in V$ and $(\overline{a}, e^{\overline{a}})$ is generic in V, i.e. $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = \dim V = n$.

Remark

The result implies many cases of Zilber's Conjecture

Polynomials with arbitrary iterations of exponentiations

Remark

Let $p(x_1, y_1, \ldots, y_n) \in \mathbb{Q}^{alg}[x_1, y_1, \ldots, y_n]$ a nonzero irreducible polynomial depending on x and the last variable y_n. Let $p(z, e^z, e^{e^z}, \ldots, e^{e^{e^{\cdots^{e^z}}}}) = 0$, the corrisponding system in $2n$ variables is:

$$V = \begin{cases} p(x_1, y_1, \ldots, y_n) = 0 \\ x_{i+1} = y_i \end{cases} \tag{8}$$

for all $i = 1, \ldots, n-1$. V is a variety of $\dim V = n$.

1. There exists a solution is unconditionally and no restriction on the coefficients, just avoid polynomials of the form $p(x, y_1, \ldots, y_n) = g(x) y_1^{k_1} \cdot \ldots \cdot y_n^{k_n}$, where $g(x) \in \mathbb{C}[x]$.
2. There is a generic solution modulo (SC)

Proof of Theorem DFT

Existence of solutions:

Theorem (Masser Brownawell-DFT '17)

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an irreducible algebraic variety such that the projection onto the first coordinates $\pi_1(V)$ is Zariski dense in $\mathbb{C}^n$. Then the set $\{\bar{a} \in \mathbb{C}^n : (\bar{a}, e^{\bar{a}}) \in V\}$ is Zariski dense in $\mathbb{C}^n$.

Remark

This is unconditionally, no use of Schanuel's conjecture, no restriction on the parameters defining V.

Proof of Theorem DFT

Existence of generic solutions:

1. **Schanuel's conjecture**
2. **Masser's system**

Proof of genericity

Let $(\overline{a}, e^{\overline{a}}) \in V$ and suppose that $t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n$, i.e. it is not generic.

Without loss of generality we can assume

$$|V_{\overline{a}}| < \infty \text{ and } |V^{e^{\overline{a}}}| < \infty$$

By Schanuel's Conjecture

$$l.d.(\overline{a}) \leq t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = m < n.$$

So, there exists $M \in \mathbb{Z}^n \times \mathbb{Z}^{n-m}$ such that $M \cdot \overline{a} = 0$.

Applying exponentiation we have the relation $e^{\overline{a}^M} = 1$. The above relations define:

$$L_M = \{\overline{x} : M \cdot \overline{x} = 0\} \text{ and } T_M = \{\overline{y} : \overline{y}^M = 1\}.$$

It turns out that L_M is tangent to T_M in $\overline{1}$ and so

$$m = \dim L_M = \dim T_M.$$

Moreover, $|V_{\overline{a}}|$ and $|V^{e^{\overline{a}}}| < \infty$ imply

$e^{\overline{a}}$ is algebraic over $\mathbb{Q}(\overline{a})$, and $\overline{a}$ is algebraic over $\mathbb{Q}(e^{\overline{a}})$, i.e.

$$m = t.d._{\mathbb{Q}}(\overline{a}) = t.d._{\mathbb{Q}}(\overline{a}, e^{\overline{a}}) = t.d._{\mathbb{Q}}(e^{\overline{a}}).$$

Proof of genericity

In other words $\overline{a}$ is generic in L_M and $e^{\overline{a}}$ is generic in T_M.
We consider

$$W_N = \{(\overline{x}, \overline{y}) \in V : \overline{x} \in L_N \wedge \mid V_{\overline{x}} \mid < \infty \wedge \mid V^{\overline{y}} \mid < \infty\},$$

where $N \in \mathbb{C}^n \times \mathbb{C}^{n-m}$. If $N = M$ then $(\overline{a}, e^{\overline{a}}) \in W_M$.
$\mid V_{\overline{a}} \mid < \infty$ implies

$$\dim W_M = \dim \pi_1(W_M) \leq \dim L_M.$$

Moreover $(\overline{a}, e^{\overline{a}}) \in W_M$ so, $\dim W_M \geq \dim L_M$. We have

$$\dim W_M = \dim L_M.$$

Let W'_M be the irreducible components of the Zariski closure of W_M containing the point $(\overline{a}, e^{\overline{a}})$.

Since $(\overline{a}, e^{\overline{a}}) \in W'_M$ is generic and $e^{\overline{a}} \in \pi_2(W'_M)$ then $\pi_2(W'_M) \subseteq T_M$, so its Zariski closure is contained in T_M.

Moreover, $e^{\overline{a}}$ is generic in T_M and $e^{\overline{a}} \in \pi_2(W'_M)$, then we have

$$T_M = \overline{\pi_2(W'_M)}^{Zar}.$$

Proof of genericity

We consider $\mathcal{U} = \{\overline{\pi_2(W'_M)}^{Zar} : \overline{\pi_2(W'_M)}^{Zar} = T_M\}$

Since $\mathcal{U}$ is a countable definable family in $(\mathbb{C}^*)^n$, and $\mathbb{C}$ is ω_1-saturated then $\mathcal{U}$ is either finite or co-countable, and since it is countable then $\mathcal{U}$ is necessarily finite, i.e. $\mathcal{U} = \{H_1, \dots, H_l\}$. So, $T_M = H_i$, for some $i = 1, \dots, l$.
We can avoid such tori by adding finitely many inequalities in the Masser's system which guarantees that the solution is a generic. □

Next goals:

1) Eliminate the hypothesis that V is defined over $\mathbb{Q}^{alg}$
(work going on with Fornasiero, Gunaydin and Terzo)

2) Weaken the hypothesis that $\pi_1(V)$ and $\pi_2(V)$ are dominant.