

Relational Proof Interpretations

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Logic Colloquium

Udine, 23 July 2018

Thanks to collaborators: Martín Escardó, Thomas Powell, Gilda Ferreira, Jaime Gaspar, Dan Hernest

proof interpretations
(Dialectica, realizability,...)

classical
proof

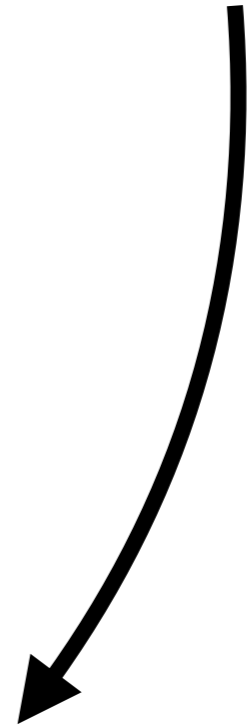
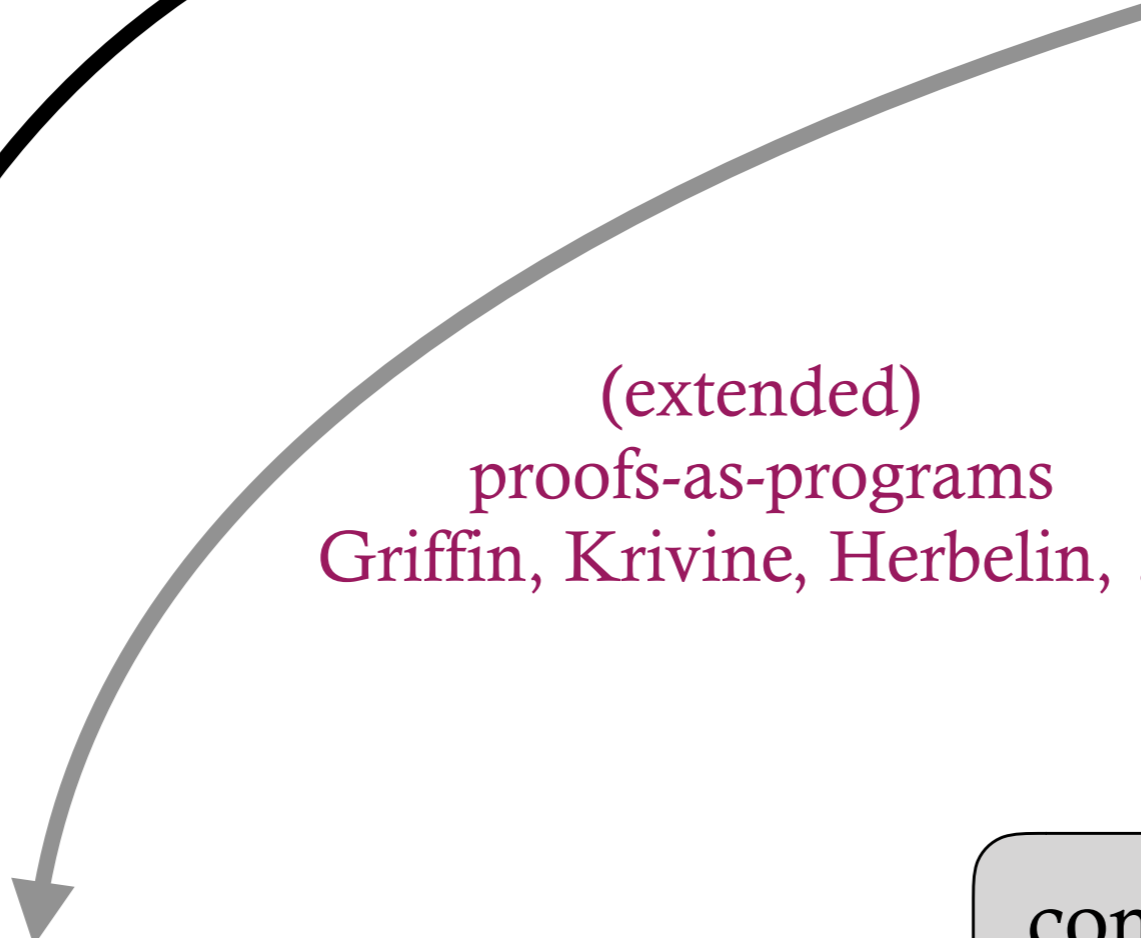
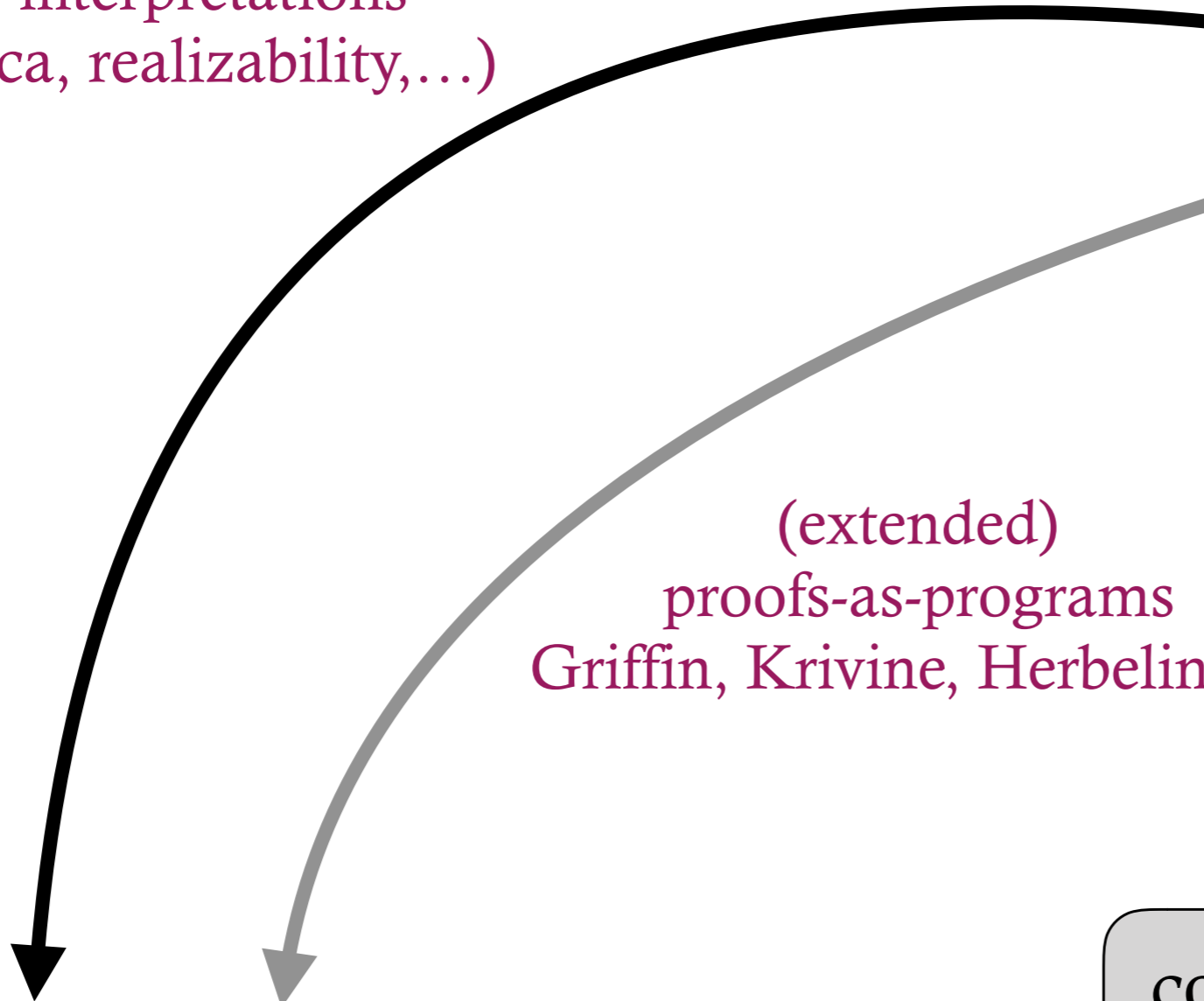
(extended)
proofs-as-programs
Griffin, Krivine, Herbelin, ...

Brouwer,
Bishop,
Bridges,...

computer
programs

constructive
proof

proofs-as-programs



Plan

- **Part 1: Sets vs Relations**
(*intuitionistic logic*)

relational approach
is more general

- **Part 2: Unification**
(*linear logic*)

interpretations (only)
differ in treatment of $!A$

- **Part 3: Games and Applications**
(*classical logic*)

higher-order games explain
higher-order programs

Part 1:

Sets vs Relations

(realizability vs dialectica)

ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

$$n \text{ r } (s = t) \quad \equiv \quad (n = 0) \wedge (s = t)$$

$$n \text{ r } A \wedge B \quad \equiv \quad n_0 \text{ r } A \wedge n_1 \text{ r } B$$

$$n \text{ r } A \vee B \quad \equiv \quad (n_0 = 0 \wedge n_1 \text{ r } A) \vee (n_0 \neq 1 \wedge n_1 \text{ r } B)$$

$$n \text{ r } A \rightarrow B \quad \equiv \quad \forall a (a \text{ r } A \rightarrow \{n\}(a) \downarrow \wedge \{n\}(a) \text{ r } B)$$

$$n \text{ r } \exists z A(z) \quad \equiv \quad n_1 \text{ r } A(n_0)$$

$$n \text{ r } \forall z A(z) \quad \equiv \quad \forall x (\{n\}(x) \downarrow \wedge \{n\}(x) \text{ r } A(x))$$

Theorem (Kleene-Nelson).

If $\text{HA} \vdash A$ then $\text{HA} \vdash n \text{ r } A$, for some numeral n

Realizability

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \underbrace{\{ n : n r A \}}_{\text{set of realizers of } A}$$

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$|A \wedge B|_{y,w}^{x,v} \equiv |A|_y^x \wedge |B|_w^v$$

$$|A \vee B|_{y,w}^{x,v,b} \equiv (b=0 \wedge |A|_y^x) \vee (b \neq 1 \wedge |B|_w^v)$$

$$|A \rightarrow B|_{x,w}^{f,g} \equiv |A|_{g(x,w)}^x \rightarrow |B|_w^{f(x)}$$

$$|\forall z A(z)|_{y,s}^f \equiv |A(s)|_y^{f(s)}$$

$$|\exists z A(z)|_y^{x,s} \equiv |A(s)|_y^x$$

Theorem (Gödel).

If $HA \vdash A$ then $T \vdash \forall y |A|_y^t$ for some term $t \in T$

Dialectica Interpretation

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \{ (x, y) : \underbrace{|A|_y^x}_{\text{relation between arguments and counter-arguments}} \}$$

Example

α is eventually bounded

α is bounded

$$A \equiv \exists n \forall i \geq n (\alpha(i) \leq n) \rightarrow \exists k \forall j (\alpha(j) \leq k)$$

$$|A|_{n,j}^{f,g} \equiv (g(n,j) \geq n \rightarrow \alpha(g(n,j)) \leq n) \rightarrow \alpha(j) \leq f(n)$$

$$f(n) = \max\{n, \max\{\alpha(i) \mid i < n\}\}$$

$$g(n,j) = j$$

so... which one is better, sets or relations?

relational approach
is more general

*Realizability can also be
presented in a 'relational' style*

Kleene realizability

$$n \text{ r } A \wedge B \quad \equiv \quad n_0 \text{ r } A \wedge n_1 \text{ r } B$$

$$n \text{ r } A \vee B \quad \equiv \quad (n_0 = 0 \wedge n_1 \text{ r } A) \vee (n_0 \neq 1 \wedge n_1 \text{ r } B)$$

$$n \text{ r } A \rightarrow B \quad \equiv \quad \forall a (a \text{ r } A \rightarrow \{n\}(a) \downarrow \wedge \{n\}(a) \text{ r } B)$$

$$n \text{ r } \exists z A(z) \quad \equiv \quad n_1 \text{ r } A(n_0)$$

$$n \text{ r } \forall z A(z) \quad \equiv \quad \forall x (\{n\}(x) \downarrow \wedge \{n\}(x) \text{ r } A(x))$$

Relational presentation

$$|A \wedge B|_a^n \quad \equiv \quad |A|_{a_0}^{n_0} \wedge |B|_{a_1}^{n_1}$$

$$|A \vee B|_a^n \quad \equiv \quad (n_0 = 0 \wedge |A|_a^{n_1}) \vee (n_0 \neq 1 \wedge |B|_a^{n_1})$$

$$|A \rightarrow B|_a^n \quad \equiv \quad \forall b |A|_b^{a_0} \rightarrow (\{n\}(a_0) \downarrow \wedge |B|_{a_1}^{\{n\}(a_0)})$$

$$|\exists z A(z)|_a^n \quad \equiv \quad |A(n_0)|_a^{n_1}$$

$$|\forall z A(z)|_a^n \quad \equiv \quad \{n\}(a_0) \downarrow \wedge |A(a_0)|_{a_1}^{\{n\}(a_0)}$$

$n \text{ r } A$

iff

$$\forall a |A|_a^n$$

Kreisel modified realizability

$$x, v \text{ mr } A \wedge B \quad \equiv \quad x \text{ mr } A \wedge v \text{ mr } B$$

$$x, v, b \text{ mr } A \vee B \quad \equiv \quad (b=0 \wedge x \text{ mr } A) \vee (b \neq 0 \wedge v \text{ mr } B)$$

$$f \text{ mr } A \rightarrow B \quad \equiv \quad \forall x (x \text{ mr } A \rightarrow f(x) \text{ mr } B)$$

$$x, s \text{ mr } \exists z A(z) \quad \equiv \quad x \text{ mr } A(s)$$

$$f \text{ mr } \forall z A(z) \quad \equiv \quad \forall x (f(x) \text{ mr } A(x))$$

Relational presentation

$$|A \wedge B|_{y,w}^{x,v} \quad \equiv \quad |A|_y^x \wedge |B|_w^v$$

$$|A \vee B|_{y,w}^{x,v,b} \quad \equiv \quad (b=0 \wedge |A|_y^x) \vee (b \neq 1 \wedge |B|_w^v)$$

$$|A \rightarrow B|_{x,w}^f \quad \equiv \quad \forall y |A|_y^x \rightarrow |B|_w^{f(x)}$$

$$|\forall z A(z)|_{y,s}^f \quad \equiv \quad |A(s)|_y^{f(s)}$$

$$|\exists z A(z)|_y^{x,s} \quad \equiv \quad |A(s)|_y^x$$

$x \text{ mr } A$
iff
 $\forall y |A|_y^x$

Gödel Dialectica interpretation

$$\begin{aligned} |A \wedge B|_{y,w}^{x,v} &\equiv |A|_y^x \wedge |B|_w^v \\ |A \vee B|_{y,w}^{x,v,b} &\equiv (b=0 \wedge |A|_y^x) \vee (b \neq 1 \wedge |B|_w^v) \\ |A \rightarrow B|_{x,w}^{f,g} &\equiv |A|_{g(x,w)}^x \rightarrow |B|_w^{f(x)} \\ |\forall z A(z)|_{y,s}^f &\equiv |A(s)|_y^{f(s)} \\ |\exists z A(z)|_y^{x,s} &\equiv |A(s)|_y^x \end{aligned}$$

$$A^D(x,y) \text{ iff } |A|_y^x$$

Kreisel modified realizability

$$\begin{aligned} |A \wedge B|_{y,w}^{x,v} &\equiv |A|_y^x \wedge |B|_w^v \\ |A \vee B|_{y,w}^{x,v,b} &\equiv (b=0 \wedge |A|_y^x) \vee (b \neq 1 \wedge |B|_w^v) \\ |A \rightarrow B|_{x,w}^f &\equiv \forall y |A|_y^x \rightarrow |B|_w^{f(x)} \\ |\forall z A(z)|_{y,s}^f &\equiv |A(s)|_y^{f(s)} \\ |\exists z A(z)|_y^{x,s} &\equiv |A(s)|_y^x \end{aligned}$$

$$x \text{ mr } A \text{ iff } \forall y |A|_y^x$$



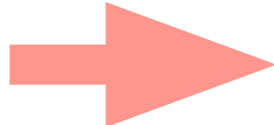
Part 2:

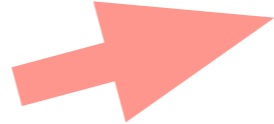
Linear Logic

(it's all about the bang!)

Linear Logic

A refinement of classical and intuitionistic logic

$A \rightarrow B$  $!A \multimap B$

$A \wedge B$ 

$A \& B$



$A \otimes B$

call-by-name translation

$$(A \wedge B)^* \equiv A^* \& B^*$$

$$(A \vee B)^* \equiv !A^* \oplus !B^*$$

$$(A \rightarrow B)^* \equiv !A^* \multimap B^*$$

$$(\forall z A)^* \equiv \forall z A^*$$

$$(\exists z A)^* \equiv \exists z !A^*$$

call-by-value translation

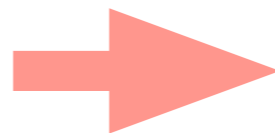
$$(A \wedge B)^\circ \equiv A^\circ \otimes B^\circ$$

$$(A \vee B)^\circ \equiv A^\circ \oplus B^\circ$$

$$(A \rightarrow B)^\circ \equiv !(A^\circ \multimap B^\circ)$$

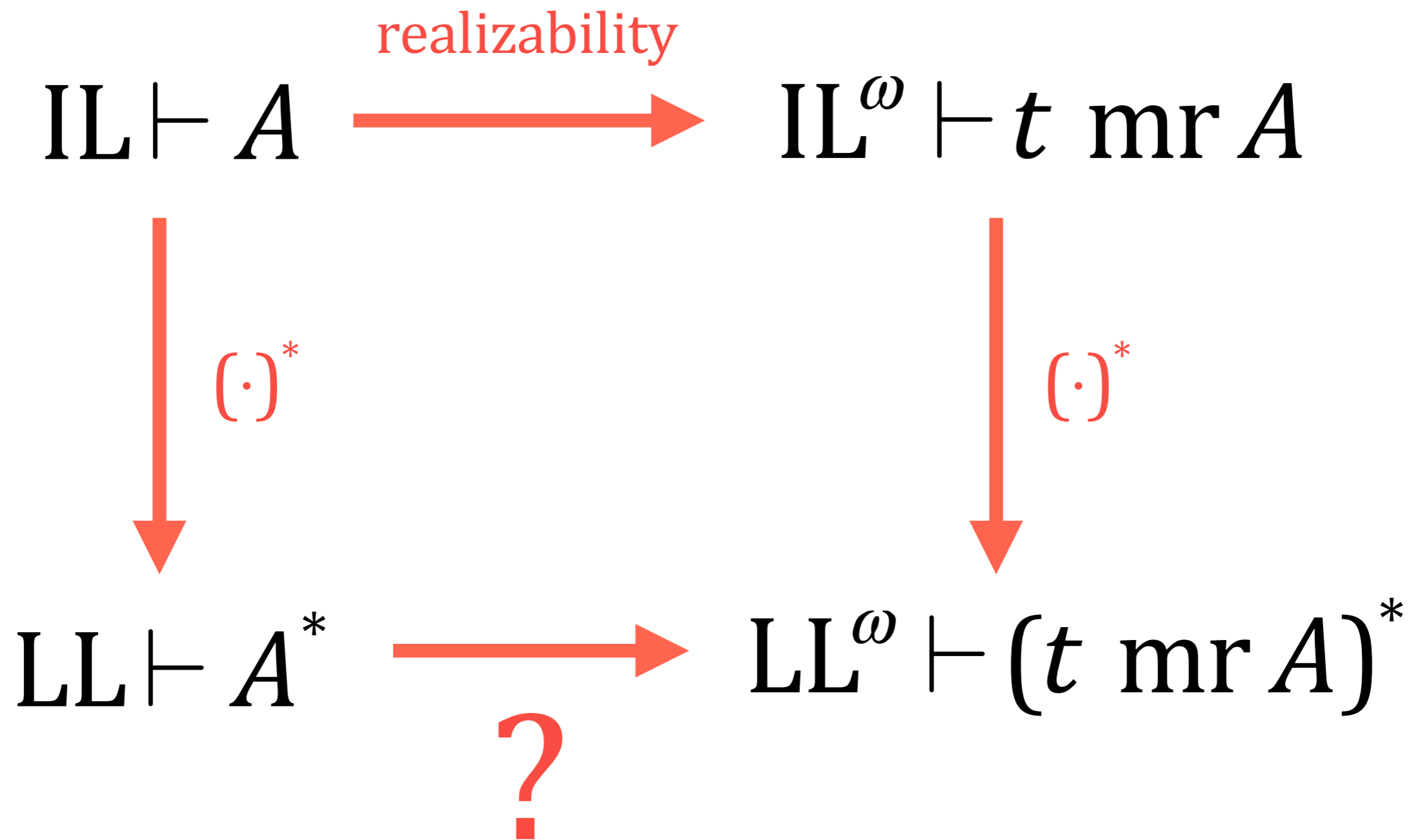
$$(\forall z A)^\circ \equiv !\forall z A^\circ$$

$$(\exists z A)^\circ \equiv \exists z A^\circ$$

$$IL \vdash A$$


$$LL \vdash A^\circ$$

$$LL \vdash A^*$$



Interpretation of Linear Logic

$$|A \otimes B|_{y,w}^{x,v} \equiv |A|_y^x \otimes |B|_w^v$$

$$|A \oplus B|_{y,w}^{x,v,b} \equiv (b=0 \ \& \ |A|_y^x) \oplus (b \neq 1 \ \& \ |B|_w^v)$$

$$|A \& B|_{y,w,b}^{x,v} \equiv (b=0 \ \& \ |A|_y^x) \oplus (b \neq 1 \ \& \ |B|_w^v)$$


$$|A \multimap B|_{x,w}^{f,g} \equiv |A|_{g(x,w)}^x \multimap |B|_w^{f(x)}$$

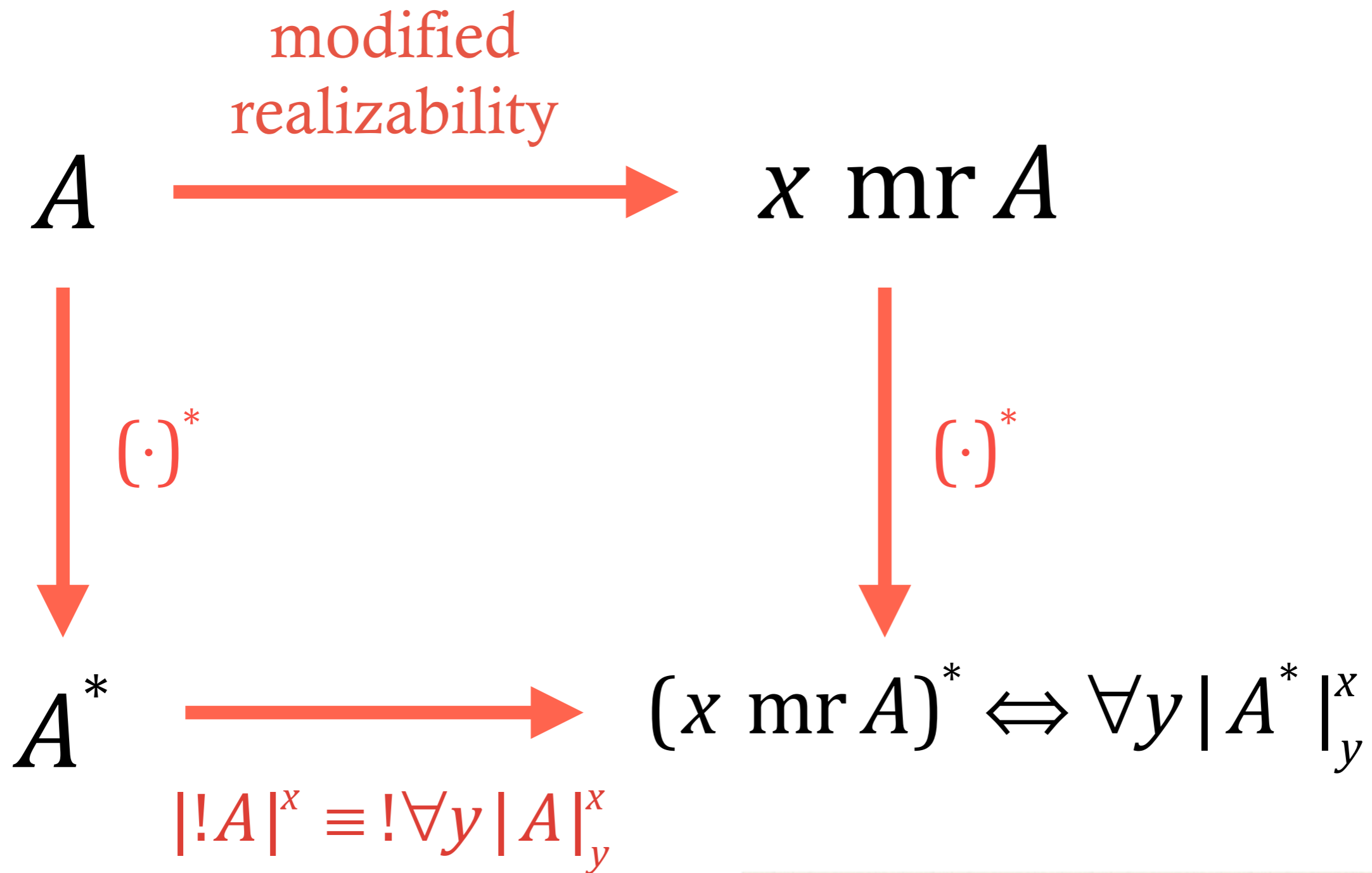
$$|\forall z A(z)|_{y,s}^f \equiv |A(s)|_y^{f(s)}$$

$$|\exists z A(z)|_y^{x,s} \equiv |A(s)|_y^x$$

based on earlier work of
de Paiva and Shirahata

 P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

 G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**,
Logical Methods in Computer Science, 7(1), 2011



interpretations (only)
differ in treatment of $!A$

!A	Trans.	Interpretation
$ A ^x \equiv !\forall y A _y^x$	$(\cdot)^*$ or $(\cdot)^\circ$	Kreisel modified realizability
$ A _a^x \equiv !\forall y \in a A _y^x$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$ A _a^x \equiv ! A _a^x$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$ A ^x \equiv !\forall y A _y^x \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$ A ^x \equiv !\forall y A _y^x \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$ A _a^x \equiv !\forall y \in a A _y^x \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

Part 3:

Applications

(classical logic and games)

How about classical logic,
arithmetic and analysis?

law of excluded middle

$$A \vee \neg A$$

double negation elimination

$$\neg\neg A \rightarrow A$$

pre-linearity

$$(A \rightarrow B) \vee (B \rightarrow A)$$

Peirce's law

$$((A \rightarrow B) \rightarrow B) \rightarrow A$$

Markov principle

$$\neg \forall n D(n) \rightarrow \exists n \neg D(n)$$

Drinker's paradox

$$\exists x (D(x) \rightarrow \forall y D(y))$$

finite choice

$$\forall n < k \exists i A(n, i) \rightarrow \exists s \forall n < k A(n, s_n)$$

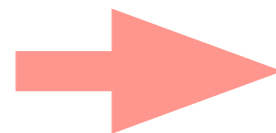
Gödel-Gentzen translation

$$\begin{aligned}(P)^G &\equiv \neg\neg P \\ (A \wedge B)^G &\equiv A^G \wedge B^G \\ (A \vee B)^G &\equiv \neg\neg(A^G \vee B^G) \\ (A \rightarrow B)^G &\equiv A^G \rightarrow B^G \\ (\forall z A)^G &\equiv \forall z A^G \\ (\exists z A)^G &\equiv \neg\neg\exists z A^G\end{aligned}$$

Kuroda translation


$$\begin{aligned}(P)^K &\equiv P \\ (A \wedge B)^K &\equiv A^K \wedge B^K \\ (A \vee B)^K &\equiv A^K \vee B^K \\ (A \rightarrow B)^K &\equiv A^K \rightarrow B^K \\ (\forall z A)^K &\equiv \forall z \neg\neg A^K \\ (\exists z A)^K &\equiv \exists z A^K\end{aligned}$$

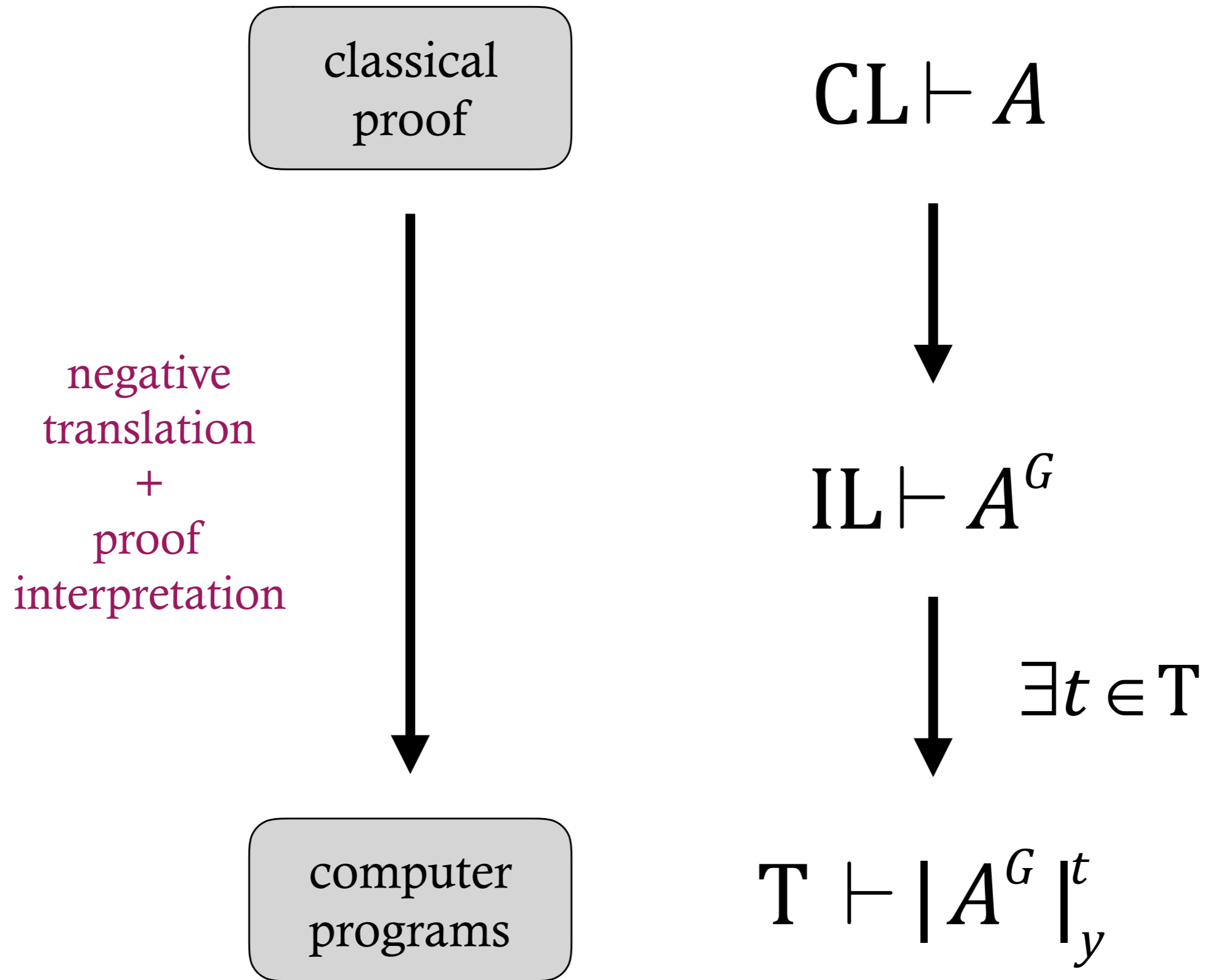
$\text{CL} \vdash A$



$\text{IL} \vdash A^G$

$\text{IL} \vdash \neg\neg A^K$

 G. Ferreira and P. Oliva, **On the relation between various negative translations**, Logic, Construction, Computation, vol 3, 227-258, 2012



classical
proof

$$\exists x^X \forall y^R A(x, y)$$



$$\neg \neg \exists x^X \forall y^R A(x, y)$$



program $\phi^{(X \rightarrow R) \rightarrow X}$?



$$\forall p^{X \rightarrow R} A(\phi(p), p(\phi(p)))$$

negative
translation
+
proof
interpretation



computer
programs

player
program $\phi^{(X \rightarrow R) \rightarrow X}$

higher-order games explain
higher-order programs

move **outcome**


$$\exists x^X \forall y^R A(x, y)$$

**x is a good move
given outcome y**


game continuation

$$\forall p^{X \rightarrow R} A(\phi(p), p(\phi(p)))$$

**optimal
move** **optimal
outcome**

 M. Escardó and P. Oliva, **Sequential games and optimal strategies**,
Proc. of the Royal Society A, 467:1519-1545, 2011

Logical form	Specifies
$\exists x^X \forall y^R A(x, y)$	player
$\forall n \exists x^X \forall y^R A_n(x, y)$	sequence of players
finite choice (bounded collection)	finite game
countable choice	unbounded game

 M. Escardó and P. Oliva, **Selection functions, bar recursion, and backward induction**, MSCS, 20 (2), pp .127-168, 2010

 P. Oliva and T. Powell, **A game-theoretic computational interpretation of proofs in classical analysis**, Gentzen's Centenary, 501-531, 2015


Summary


- Realizability also has a “**relational**” **presentation**
- Relational presentation allows for **interpretation of LL and unification** (including truth variants)
- Classical proofs dealt with by combining interpretation with a **negative translation**
- Classical proof (and higher-order programs) can be “explained” in terms of **higher-order games**


 P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006


 P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

 J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56:591-610, 2010

 M. Escardó and P. Oliva, **Selection functions, bar recursion, and backward induction**, MSCS, 20 (2), pp .127-168, 2010

 M. Escardó and P. Oliva, **Sequential games and optimal strategies**, Proc. of the Royal Society A, 467:1519-1545, 2011

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