Stationary reflection

Spencer Unger, joint work with Yair Hayut

Tel Aviv University

July 28, 2018

Compactness

Notions of compactness for countable sets are well known.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

Compactness

Notions of compactness for countable sets are well known.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

Theorem (Infinite Ramsey's theorem)

For every function $f : [\omega]^2 \to 2$ *there are an infinite* $A \subseteq \omega$ *and* $i \in 2$ *such that for all* $n < m$ *from* A, $f(\{n, m\}) = i$.

Compactness

Notions of compactness for countable sets are well known.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

Theorem (Infinite Ramsey's theorem)

For every function $f : [\omega]^2 \to 2$ there are an infinite $A \subseteq \omega$ and $i \in 2$ such that for all $n < m$ from A, $f(\{n, m\}) = i$.

Theorem

For every cardinal λ there is an ultrafilter U on $\{x \subseteq \lambda \mid x \text{ is finite}\}$ such that for all $\alpha < \lambda$, $\{x \mid \alpha \in x\} \in U$.

Compactness at uncountable cardinals

Compactness at uncountable cardinals is one of the central topics in modern set theory.
Examples:

Theorem (Aronszajn)

There is a tree of height ω_1 with countable levels and no cofinal branch.

Compactness at uncountable cardinals

Compactness at uncountable cardinals is one of the central topics in modern set theory.
Examples:

Theorem (Aronszajn)

There is a tree of height ω_1 with countable levels and no cofinal branch.

A cardinal κ is inaccessible if κ is regular and for all $\mu < \kappa$, the size of the powerset of μ is less than κ.

Compactness at uncountable cardinals

Compactness at uncountable cardinals is one of the central topics in modern set theory.
Examples:

Theorem (Aronszajn)

There is a tree of height ω_1 with countable levels and no cofinal branch.

A cardinal κ is inaccessible if κ is regular and for all $\mu < \kappa$, the size of the powerset of μ is less than κ.
Let κ be an uncountable cardinal.

Theorem (Folklore)

If κ satisfies a higher version of the Infinite Ramsey theorem, then κ is the κ^{th} inaccessible cardinal.

Cardinals satisfying this higher version of Ramsey's theorem are called weakly compact.

Compactness at uncountable cardinals

Theorem (Mitchell-Silver)

It is consistent with ZFC that there is a weakly compact cardinal if and only if it is consistent with ZFC that ω_2 satisfies a version of König's lemma.

Compactness at uncountable cardinals

Theorem (Mitchell-Silver)

It is consistent with ZFC that there is a weakly compact cardinal if and only if it is consistent with ZFC that ω_2 satisfies a version of König's lemma.

Theorem (Solovay)

If for every cardinal λ, there is a κ-complete ultrafilter U on $\{x \subseteq \lambda \mid |x| < \kappa\}$ such that for all $\alpha < \lambda$, $\{x \mid \alpha \in x\} \in U$, then for all regular cardinals μ, the set $\{f \mid f : \alpha \to \mu \text{ for some } \alpha < \kappa\}$ has size μ.

Cardinals κ as in the hypothesis are called strongly compact.

Morals

- ω_1 satisfies the negation of many compactness properties.

Morals

- ω_1 satisfies the negation of many compactness properties.
- Some compactness properties arise as the generalization of compactness properties of ω.

Morals

- ω_1 satisfies the negation of many compactness properties.
- Some compactness properties arise as the generalization of compactness properties of ω.
- Some compactness properties are consistent at small cardinals while others imply that a given cardinal is quite large.

Morals

- ω_1 satisfies the negation of many compactness properties.
- Some compactness properties arise as the generalization of compactness properties of ω.
- Some compactness properties are consistent at small cardinals while others imply that a given cardinal is quite large.
- Some compactness properties have strong structural influence on the universe of set theory.

Motivation

One answer is that we wish to know the extent to which the usual axioms of set theory settle questions about compactness.

Motivation

One answer is that we wish to know the extent to which the usual axioms of set theory settle questions about compactness.
For example:

Question

Can one construct a version of Aronszajn's tree on ω_1 on ω_2?

The theorem of Mitchell and Silver above shows that it is impossible using the axioms of ZFC assuming the consistency of a weakly compact cardinal.
Further, the theorem shows that the weakly compact cardinal is necessary in the sense that if we have a model of ZFC with no such trees on ω_2, then there is also a model of ZFC with a weakly compact cardinal.

Cofinality

A sequence increasing sequence $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is cofinal in an ordinal δ if the set $\{\alpha_\beta \mid \beta < \gamma\}$ is unbounded in δ.
The cofinality of δ is the least ordinal γ for which there is a cofinal sequence of length γ as above.

Cofinality

A sequence increasing sequence $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is cofinal in an ordinal δ if the set $\{\alpha_\beta \mid \beta < \gamma\}$ is unbounded in δ.
The cofinality of δ is the least ordinal γ for which there is a cofinal sequence of length γ as above.
Examples/Definitions:

- $\mathrm{cf}(\omega_\omega) = \omega$ as witnessed by $\langle \omega_n \mid n < \omega \rangle$.
- If μ is a cardinal, then the next cardinal greater than μ is denoted μ^+. Using the axiom of choice, $\mathrm{cf}(\mu^+) = \mu^+$.
- A cardinal λ is regular if $\mathrm{cf}(\lambda) = \lambda$ and singular otherwise.

Stationary sets

Let μ be an ordinal.
A set $C \subseteq \mu$ is *club* in μ if it is closed (for all $\alpha < \mu$ if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$) and unbounded.

Stationary sets

Let μ be an ordinal.
A set $C \subseteq \mu$ is *club* in μ if it is closed (for all $\alpha < \mu$ if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$) and unbounded.
A set $S \subseteq \mu$ is stationary if for every club C in μ, $S \cap C$ is nonempty.
Think of stationary sets as analogous to a set of positive measure and club sets as analogous to measure one sets.

Stationary sets

Let μ be an ordinal.
A set $C \subseteq \mu$ is *club* in μ if it is closed (for all $\alpha < \mu$ if $C \cap \alpha$ is unbounded in α, then $\alpha \in C$) and unbounded.
A set $S \subseteq \mu$ is stationary if for every club C in μ, $S \cap C$ is nonempty.
Think of stationary sets as analogous to a set of positive measure and club sets as analogous to measure one sets.
Facts and examples: Let $\kappa < \lambda$ be regular cardinals.

- The club subsets of λ form a λ-complete filter.
- The set $\{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$ is stationary. We call this set S^λ_κ.

Stationary reflection

Let λ be a regular cardinal.

Definition

Let α be an ordinal with $\mathrm{cf}(\alpha) > \omega$. We say that a set $S \subseteq \lambda$ reflects at α if $S \cap \alpha$ is stationary in α.

Stationary reflection

Let λ be a regular cardinal.

Definition

Let α be an ordinal with $\mathrm{cf}(\alpha) > \omega$. We say that a set $S \subseteq \lambda$ reflects at α if $S \cap \alpha$ is stationary in α.

Definition

Stationary reflection at λ is the assertion that every stationary subset of λ reflects at some ordinal $\alpha < \lambda$.

Stationary reflection

Let λ be a regular cardinal.

Definition

Let α be an ordinal with $\mathrm{cf}(\alpha) > \omega$. We say that a set $S \subseteq \lambda$ reflects at α if $S \cap \alpha$ is stationary in α.

Definition

Stationary reflection at λ is the assertion that every stationary subset of λ reflects at some ordinal $\alpha < \lambda$.

There are variations:

- We might focus on stationary subsets of S^λ_κ for some κ.
- We might ask that collections of stationary sets reflect at a common point.

Stationary reflection

Let λ be a regular cardinal.

Definition

Let α be an ordinal with $\mathrm{cf}(\alpha) > \omega$. We say that a set $S \subseteq \lambda$ reflects at α if $S \cap \alpha$ is stationary in α.

Definition

Stationary reflection at λ is the assertion that every stationary subset of λ reflects at some ordinal $\alpha < \lambda$.

There are variations:

- We might focus on stationary subsets of S^λ_κ for some κ.
- We might ask that collections of stationary sets reflect at a common point.

A stationary set which does not reflect at any α is called *non-reflecting*.

Motivation

Recall that part of our motivation is to determine the extent to which the usual axioms of set theory settle questions about compactness.

Further if they do not we would like to measure the consistency strength in terms of large cardinals.

Motivation

Recall that part of our motivation is to determine the extent to which the usual axioms of set theory settle questions about compactness.
Further if they do not we would like to measure the consistency strength in terms of large cardinals.

Theorem (Todorcevic)

If there is a non-reflecting stationary subset of S^λ_ω, then there is a graph of size λ of chromatic number at least ω_1 all of whose subgraphs of smaller cardinality are countably chromatic.

Theorem

If there is a non-reflecting stationary subset of S^λ_ω, then there is a non-metrizable locally compact topological space all of whose smaller cardinality subspaces are metrizable.

Consistency results

Theorem (Shelah-Harrington)

It is consistent that every stationary subset of $S^{\omega_2}_{\omega}$ reflects if and only if it is consistent that there is a Mahlo cardinal.

A cardinal κ is Mahlo if the set of inaccessible cardinals below κ is stationary.

Consistency results

Theorem (Shelah-Harrington)

It is consistent that every stationary subset of $S^{\omega_2}_{\omega}$ reflects if and only if it is consistent that there is a Mahlo cardinal.

A cardinal κ is Mahlo if the set of inaccessible cardinals below κ is stationary.

Theorem (Magidor)

It is consistent that every pair of stationary subsets of $S^{\omega_2}_{\omega}$ reflect at common point if and only if it is consistent that there is a weakly compact cardinal.

Consistency results

Theorem (Magidor)

If it is consistent that there are infinitely many supercompact cardinals, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ *reflects.*

Consistency results

Theorem (Magidor)

If it is consistent that there are infinitely many supercompact cardinals, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

Theorem (Hayut,U)

If it is consistent that there is a cardinal κ which is κ^+-supercompact, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

Consistency results

Theorem (Magidor)

If it is consistent that there are infinitely many supercompact cardinals, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

Theorem (Hayut,U)

If it is consistent that there is a cardinal κ which is κ^+-supercompact, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

In recent joint work with Ben-Neria and Hayut, we use the techniques from this theorem to get a model with stationary reflection at the sucessor of a singular cardinal where the singular cardinal hypothesis fails.

Consistency results

Theorem (Magidor)

If it is consistent that there are infinitely many supercompact cardinals, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

Theorem (Hayut,U)

If it is consistent that there is a cardinal κ which is κ^+-supercompact, then it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

In recent joint work with Ben-Neria and Hayut, we use the techniques from this theorem to get a model with stationary reflection at the sucessor of a singular cardinal where the singular cardinal hypothesis fails.
For the remainder of the talk we'll aim to get a singular strong limit cardinal κ of cofinality ω where every stationary subset of κ^+ reflects.

Let κ be an uncountable cardinal.

Proposition

If there is a nonprincipal κ-complete normal ultrafilter on κ, then every stationary subset of κ reflects.

An ultrafilter U on κ is normal if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ from U , the set $\{\alpha \mid \alpha \in A_\beta \text{ for all } \beta < \alpha\} \in U$.

Let κ be an uncountable cardinal.

Proposition

If there is a nonprincipal κ-complete normal ultrafilter on κ, then every stationary subset of κ reflects.

An ultrafilter U on κ is normal if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ from U , the set $\{\alpha \mid \alpha \in A_\beta \text{ for all } \beta < \alpha\} \in U$.

Proof.

Suppose otherwise. Then there is a stationary S such that for all $\alpha < \kappa$ there is a club C_α in α which is disjoint from S.

Let κ be an uncountable cardinal.

Proposition

If there is a nonprincipal κ-complete normal ultrafilter on κ, then every stationary subset of κ reflects.

An ultrafilter U on κ is normal if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ from U , the set $\{\alpha \mid \alpha \in A_\beta \text{ for all } \beta < \alpha\} \in U$.

Proof.

Suppose otherwise. Then there is a stationary S such that for all $\alpha < \kappa$ there is a club C_α in α which is disjoint from S.
Let $C = \{\gamma \mid \{\alpha \mid \gamma \in C_\alpha\} \in U\}$. It is clear that C is closed using the κ-completeness of U. It remains to show C is unbounded.

Let κ be an uncountable cardinal.

Proposition

If there is a nonprincipal κ-complete normal ultrafilter on κ, then every stationary subset of κ reflects.

An ultrafilter U on κ is normal if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ from U , the set $\{\alpha \mid \alpha \in A_\beta$ for all $\beta < \alpha\} \in U$.

Proof.

Suppose otherwise. Then there is a stationary S such that for all $\alpha < \kappa$ there is a club C_α in α which is disjoint from S.
Let $C = \{\gamma \mid \{\alpha \mid \gamma \in C_\alpha\} \in U\}$. It is clear that C is closed using the κ-completeness of U. It remains to show C is unbounded.
If not, then for all large enough γ the set $A_\gamma = \{\alpha \mid \gamma \notin C_\alpha\} \in U$.
By normality, $\{\alpha \mid$ for all $\gamma < \alpha$, $\alpha \in A_\gamma\} \in U$.

Let κ be an uncountable cardinal.

Proposition

If there is a nonprincipal κ-complete normal ultrafilter on κ, then every stationary subset of κ reflects.

An ultrafilter U on κ is normal if for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ from U, the set $\{\alpha \mid \alpha \in A_\beta \text{ for all } \beta < \alpha\} \in U$.

Proof.

Suppose otherwise. Then there is a stationary S such that for all $\alpha < \kappa$ there is a club C_α in α which is disjoint from S.
Let $C = \{\gamma \mid \{\alpha \mid \gamma \in C_\alpha\} \in U\}$. It is clear that C is closed using the κ-completeness of U. It remains to show C is unbounded.
If not, then for all large enough γ the set $A_\gamma = \{\alpha \mid \gamma \notin C_\alpha\} \in U$.
By normality, $\{\alpha \mid \text{for all } \gamma < \alpha,\ \alpha \in A_\gamma\} \in U$.
It follows that this set is nonempty, so let α be an element. It follows that for all $\gamma < \alpha$, $\gamma \notin C_\alpha$ and so C_α is empty, a contradiction. □

Similarly, if there is a κ-complete, normal ultrafilter U on $\{x \subseteq \kappa^+ \mid |x| < \kappa\}$ such that for all $\alpha < \kappa^+$, $\{x \mid \alpha \in x\} \in U$, then every stationary subset of $S^{\kappa^+}_{<\kappa}$ reflects.

Similarly, if there is a κ-complete, normal ultrafilter U on $\{x \subseteq \kappa^+ \mid |x| < \kappa\}$ such that for all $\alpha < \kappa^+$, $\{x \mid \alpha \in x\} \in U$, then every stationary subset of $S^{\kappa^+}_{<\kappa}$ reflects.

Cardinals as in the hypothesis are called κ^+-supercompact.

How do you change cofinality to ω?

$$\langle \alpha_0, \alpha_1, \ldots \alpha_n \rangle \geq$$
$$\langle \alpha_0, \alpha_1, \ldots \alpha_n, \alpha_{n+1} \rangle$$

How do you change cofinality to ω?

$$\langle \alpha_0, \alpha_1, \ldots \alpha_n, A \rangle \geq$$
$$\langle \alpha_0, \alpha_1, \ldots \alpha_n, \alpha_{n+1}, B \rangle$$

This forcing is called Prikry forcing.

Bounded stationary reflection

Recall that if κ is regular, then $S^{\kappa^+}_{\kappa}$ does not reflect.

Recall that if κ is regular, then $S^{\kappa^+}_{\kappa}$ does not reflect.

Theorem (Cummings-Foreman-Magidor)

If κ is κ^+-supercompact, then in the extension by Prikry forcing every stationary subset of $\kappa^+ \setminus (S^{\kappa^+}_{\kappa})^V$ reflects.

Bounded stationary reflection

Recall that if κ is regular, then $S^{\kappa^+}_\kappa$ does not reflect.

Theorem (Cummings-Foreman-Magidor)

If κ is κ^+-supercompact, then in the extension by Prikry forcing every stationary subset of $\kappa^+ \setminus (S^{\kappa^+}_\kappa)^V$ reflects.

There are forcings to destroy the stationarity of nonreflecting sets, but it is not clear that destroying the stationarity of $(S^{\kappa^+}_\kappa)^V$ will give full stationary reflection.

Ideas from the proof

Work over iterated ultrapowers. Let U be a normal measure on κ.

- We can form the ultrapower of V by U and derive an associated elementary embedding $j : V \to M$ where M is the transitive collapse of the ultrapower.

Ideas from the proof

Work over iterated ultrapowers. Let U be a normal measure on κ.

- We can form the ultrapower of V by U and derive an associated elementary embedding $j : V \to M$ where M is the transitive collapse of the ultrapower.
- We can iterate this procedure to obtain a sequence of ultrapowers M_n for $n \in \omega$ with elementary maps between them. In particular, we have $j_n : V \to M_n$.

Ideas from the proof

Work over iterated ultrapowers. Let U be a normal measure on κ.

- We can form the ultrapower of V by U and derive an associated elementary embedding $j : V \to M$ where M is the transitive collapse of the ultrapower.
- We can iterate this procedure to obtain a sequence of ultrapowers M_n for $n \in \omega$ with elementary maps between them. In particular, we have $j_n : V \to M_n$.
- Commutativity of the embeddings between different M_n gives rise to a direct limit model M_ω.

Ideas from the proof

Work over iterated ultrapowers. Let U be a normal measure on κ.

- We can form the ultrapower of V by U and derive an associated elementary embedding $j : V \to M$ where M is the transitive collapse of the ultrapower.
- We can iterate this procedure to obtain a sequence of ultrapowers M_n for $n \in \omega$ with elementary maps between them. In particular, we have $j_n : V \to M_n$.
- Commutativity of the embeddings between different M_n gives rise to a direct limit model M_ω.
- The sequence $\langle j_n(\kappa) \mid n < \omega \rangle$ is Prikry generic over M_ω.

Work over iterated ultrapowers. Let U be a normal measure on κ.

- We can form the ultrapower of V by U and derive an associated elementary embedding $j : V \to M$ where M is the transitive collapse of the ultrapower.
- We can iterate this procedure to obtain a sequence of ultrapowers M_n for $n \in \omega$ with elementary maps between them. In particular, we have $j_n : V \to M_n$.
- Commutativity of the embeddings between different M_n gives rise to a direct limit model M_ω.
- The sequence $\langle j_n(\kappa) \mid n < \omega \rangle$ is Prikry generic over M_ω.

Theorem (Dehornoy,Bukovsky)

$M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle] = \bigcap_{n<\omega} M_n$.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.
- Let H be generic for $\mathbb{Q}$ over V and let $j_{n,\omega} \colon M_n \to M_\omega$ appropriate elementary embedding.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.
- Let H be generic for $\mathbb{Q}$ over V and let $j_{n,\omega} \colon M_n \to M_\omega$ appropriate elementary embedding.
- Suppose that S is a stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid m < \omega\rangle][H]$.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.
- Let H be generic for $\mathbb{Q}$ over V and let $j_{n,\omega} \colon M_n \to M_\omega$ appropriate elementary embedding.
- Suppose that S is a stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid m < \omega \rangle][H]$.
- Then $S \in M_n[H]$ and hence so is $S_n = \{\alpha < j_n(\kappa^+) \mid j_{n,\omega}(\alpha) \in S\}$.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.
- Let H be generic for $\mathbb{Q}$ over V and let $j_{n,\omega} \colon M_n \to M_\omega$ appropriate elementary embedding.
- Suppose that S is a stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid m < \omega\rangle][H]$.
- Then $S \in M_n[H]$ and hence so is $S_n = \{\alpha < j_n(\kappa^+) \mid j_{n,\omega}(\alpha) \in S\}$.
- If S_n is stationary, then it reflects using an indestructible version of bounded stationary reflection. Hence S reflects also.

- If we let $\mathbb{Q}$ be the forcing to destroy the "bad" stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle]$, then as a poset over V, $\mathbb{Q}$ is equivalent to adding a Cohen subset of κ^+.
- Let H be generic for $\mathbb{Q}$ over V and let $j_{n,\omega}\colon M_n \to M_\omega$ appropriate elementary embedding.
- Suppose that S is a stationary subset of $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid m < \omega\rangle][H]$.
- Then $S \in M_n[H]$ and hence so is $S_n = \{\alpha < j_n(\kappa^+) \mid j_{n,\omega}(\alpha) \in S\}$.
- If S_n is stationary, then it reflects using an indestructible version of bounded stationary reflection. Hence S reflects also.

- So we can assume that each S_n is non-stationary as witnessed by a club C_n. We can then argue that $C = \bigcap_{m<\omega} j_{n,\omega}(C_n)$ is a club disjoint from S.

- So we can assume that each S_n is non-stationary as witnessed by a club C_n. We can then argue that $C = \bigcap_{m<\omega} j_{n,\omega}(C_n)$ is a club disjoint from S.
- Unfortunately, C is not in $M_\omega[\langle j_n(\kappa) \mid m < \omega \rangle][H]$. It is only in $M_\omega[\langle j_n(\kappa) \mid m < \omega \rangle][H][\overline{\langle j_{n,\omega} ``H} \mid n < \omega \rangle]$.

- So we can assume that each S_n is non-stationary as witnessed by a club C_n. We can then argue that $C = \bigcap_{m<\omega} j_{n,\omega}(C_n)$ is a club disjoint from S.
- Unfortunately, C is not in $M_\omega[\langle j_n(\kappa) \mid m < \omega\rangle][H]$. It is only in $M_\omega[\langle j_n(\kappa) \mid m < \omega\rangle][H][\langle \overline{j_{n,\omega} ``H} \mid n < \omega\rangle]$.

Theorem

Let $\mathcal{H}$ be the sequence $\langle \overline{j_{n,\omega} ``H} \mid n < \omega\rangle$. Then $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ and stationary reflection holds at $j_\omega(\kappa^+)$ in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.
- $\mathcal{H}_n$ is equivalent to adding a Cohen subset of $j_n(\kappa^+)$ over M_n, so we can assume that $M_n[\mathcal{H}_n]$ satisfies bounded stationary reflection.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.
- $\mathcal{H}_n$ is equivalent to adding a Cohen subset of $j_n(\kappa^+)$ over M_n, so we can assume that $M_n[\mathcal{H}_n]$ satisfies bounded stationary reflection.
- If S is stationary set in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$, then it consists of points of some fixed cofinality below $j_\omega(\kappa)$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.
- $\mathcal{H}_n$ is equivalent to adding a Cohen subset of $j_n(\kappa^+)$ over M_n, so we can assume that $M_n[\mathcal{H}_n]$ satisfies bounded stationary reflection.
- If S is stationary set in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$, then it consists of points of some fixed cofinality below $j_\omega(\kappa)$.
- For each n, S can be pulled back to a set S_n of bounded cofinality in $M_n[\mathcal{H}_n]$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.
- $\mathcal{H}_n$ is equivalent to adding a Cohen subset of $j_n(\kappa^+)$ over M_n, so we can assume that $M_n[\mathcal{H}_n]$ satisfies bounded stationary reflection.
- If S is stationary set in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$, then it consists of points of some fixed cofinality below $j_\omega(\kappa)$.
- For each n, S can be pulled back to a set S_n of bounded cofinality in $M_n[\mathcal{H}_n]$.
- If S does not reflect, then each of the S_n will be nonstationary as witnessed by a club C_n using bounded stationary reflection in $M_n[\mathcal{H}_n]$.

Why?

- $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ is a Prikry type extension of M_ω.
- There are generics $\mathcal{H}_n$ over M_n such that $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}] = \bigcap_{n<\omega} M_n[\mathcal{H}_n]$.
- It follows that $H \in M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$.
- $\mathcal{H}_n$ is equivalent to adding a Cohen subset of $j_n(\kappa^+)$ over M_n, so we can assume that $M_n[\mathcal{H}_n]$ satisfies bounded stationary reflection.
- If S is stationary set in $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$, then it consists of points of some fixed cofinality below $j_\omega(\kappa)$.
- For each n, S can be pulled back to a set S_n of bounded cofinality in $M_n[\mathcal{H}_n]$.
- If S does not reflect, then each of the S_n will be nonstationary as witnessed by a club C_n using bounded stationary reflection in $M_n[\mathcal{H}_n]$.
- It follows that pushing the C_n forward to $M_\omega[\langle j_n(\kappa) \mid n < \omega\rangle][\mathcal{H}]$ and taking the intersection gives a club disjoint from S, which finishes the proof.

Question

Is it consistent that there is a singular cardinal κ of uncountable cofinality where the singular cardinal hypothesis fails and every stationary subset of κ^+ reflects?