On the Structure of the Wadge degrees of BQO-valued Borel functions

Takayuki Kihara

Nagoya University, Japan

Joint Work With

Antonio Montalbán (UC Berkeley)

Logic Colloquium, Udine, Italy, July 23, 2018

- The structure of Wadge degrees of Borel subsets of Baire space (equivalently Borel functions $f : \omega^\omega \to \mathbf{2}$) is very simple.
- If we change the domain to non-zero-dimensional spaces (e.g. $\mathbb{R}$), the Wadge degree structure can become very complicated (Ikegami, Schlicht, Tanaka, and others)

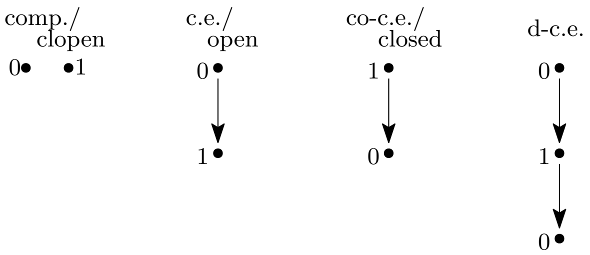
- The structure of Wadge degrees of Borel subsets of Baire space (equivalently Borel functions $f : \omega^\omega \to \mathbf{2}$) is very simple.
- If we change the domain to non-zero-dimensional spaces (e.g. $\mathbb{R}$), the Wadge degree structure can become very complicated (Ikegami, Schlicht, Tanaka, and others)
- What happens if we change the codomain?
- (van Engelen, Miller, and Steel 1987) If the codomain $\mathcal{Q}$ is better-quasi-ordered (BQO), then the Wadge degrees of Borel functions $f : \omega^\omega \to \mathcal{Q}$ is still BQO.

- The structure of Wadge degrees of Borel subsets of Baire space (equivalently Borel functions $f : \omega^\omega \to \mathbf{2}$) is very simple.
- If we change the domain to non-zero-dimensional spaces (e.g. $\mathbb{R}$), the Wadge degree structure can become very complicated (Ikegami, Schlicht, Tanaka, and others)
- What happens if we change the codomain?
- (van Engelen, Miller, and Steel 1987) If the codomain $\mathcal{Q}$ is better-quasi-ordered (BQO), then the Wadge degrees of Borel functions $f : \omega^\omega \to \mathcal{Q}$ is still BQO.
- (Main Theorem) It is not only BQO, but also has a very simple description.
- Our proof involves various notions from computability theory; however it makes the whole proof extremely simple (compared to previous works on Wadge degrees on Borel sets/functions!)

Key Observation at the First Level

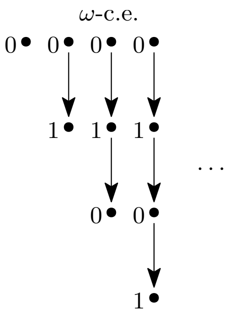
$\mathbf{\Delta^0_2}$-procedures are represented by effective approximation procedures on well-founded trees (≈ mind-change counters).

Tree/Forest-representation of various Δ^0_2 sets:



- (computable/clopen) Given an input x, effectively decide $x \notin A$ (indicated by 0) or $x \in A$ (indicated by 1).
- (c.e./open) Given an input x, begin with $x \notin A$ (indicated by 0) and later x can be enumerated into A (indicated by 1).
- (co-c.e./closed) Given an input x, begin with $x \in A$ (indicated by 1) and later x can be removed from A (indicated by 0).
- (d-c.e.) Begin with $x \notin A$ (indicated by 0), later x can be enumerated into A (indicated by 1), and x can be removed from A again (indicated by 0).

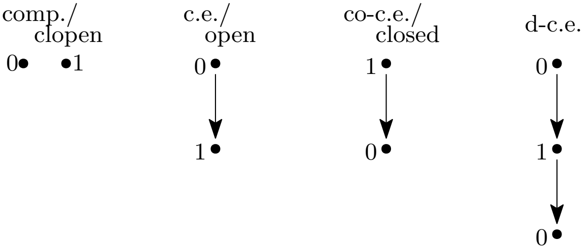
Forest-representation of a complete ω-c.e. set:



(ω-c.e.) The representation of "ω-c.e." is a forest consists of linear orders of finite length (a linear order of length $n+1$ represents "n-c.e.").

- Given an input x, effectively choose a number $n \in \omega$ giving a bound of the number of times of mind-changes until deciding $x \in A$.

Tree/Forest-representation of various $\mathbf{\Delta}^0_2$ sets:



Remark

- Let I_α be the $\{0,1\}$-labeled forest corresponding to the α-th level of the Hausdorff difference hierarchy (or equivalently the Ershov hierarchy).
- A set $A \subseteq \omega^\omega$ is of rank α in the Hausdorff difference hierarchy iff there is a continuous $f_A : \omega^\omega \to I_\alpha$ s.t. $A(x) = [\text{label of } f_A(x)]$ for all x, where I_α is topologized by the standard non-Hausdorff order topology.
- A set $A \subseteq \omega$ is of rank α in the Ershov hierarchy iff there is a computable $f_A : \omega \to I_\alpha$ s.t. $A(x) = [\text{label of } f_A(x)]$ for all x, where I_α is represented by the standard Sierpiński-like representation.

(Wadge) Let S be a space, and $\mathcal{Q}$ be a preorder.

- $f : S \to \mathcal{Q}$ is Wadge reducible to $g : S \to \mathcal{Q}$ (written $f \leq_W g$) if there is a continuous $\theta : S \to S$ s.t. $f(x) \leq_{\mathcal{Q}} g \circ \theta(x)$ for all $x \in S$.

(Wadge) Let S be a space, and $\mathcal{Q}$ be a preorder.

- $f : S \to \mathcal{Q}$ is Wadge reducible to $g : S \to \mathcal{Q}$ (written $f \leq_W g$) if there is a continuous $\theta : S \to S$ s.t. $f(x) \leq_{\mathcal{Q}} g \circ \theta(x)$ for all $x \in S$.

Homomorphic Quasi-Ordering (Hertling, Selivanov, and others)

$\mathcal{A} = (A, \leq_A, c_A)$, $\mathcal{B} = (B, \leq_B, c_A)$: $\mathcal{Q}$-labeled preorders (i.e. $c_X : X \to \mathcal{Q}$)
A morphism $h : \mathcal{A} \to \mathcal{B}$ is a function $h : A \to B$ s.t.

- $x \leq_A y \implies h(x) \leq_B h(y)$.
- $c_A(x) \leq_{\mathcal{Q}} c_B(h(x))$.

Write $\mathcal{A} \leq_h \mathcal{B}$ if there is a morphism $h : \mathcal{A} \to \mathcal{B}$.
The quasi-ordering $\leq_h$ is called a homomorphic quasi-order.

(Wadge) Let S be a space, and Q be a preorder.

- $f : S \to Q$ is Wadge reducible to $g : S \to Q$ (written $f \leq_W g$) if there is a continuous $\theta : S \to S$ s.t. $f(x) \leq_Q g \circ \theta(x)$ for all $x \in S$.

Homomorphic Quasi-Ordering (Hertling, Selivanov, and others)

$\mathcal{A} = (A, \leq_A, c_A)$, $\mathcal{B} = (B, \leq_B, c_A)$: Q-labeled preorders (i.e. $c_X : X \to Q$)
A morphism $h : \mathcal{A} \to \mathcal{B}$ is a function $h : A \to B$ s.t.

- $x \leq_A y \implies h(x) \leq_B h(y)$.
- $c_A(x) \leq_Q c_B(h(x))$.

Write $\mathcal{A} \leq_h \mathcal{B}$ if there is a morphism $h : \mathcal{A} \to \mathcal{B}$.
The quasi-ordering $\leq_h$ is called a homomorphic quasi-order.

Theorem (Selivanov 2007)

If k is a finite discrete order, the Wadge degrees of k-valued $\underset{\sim}{\mathbf{\Delta}}^0_2$ functions are isomorphic to the quotient homomorphic quasiorder on countable well-founded k-labeled forests.

- $\mathbf{Tree}(Q)$: The set of all countable well-founded Q-valued trees.
- $^{\sqcup}\mathbf{Tree}(Q)$: The set of all countable well-founded Q-valued forests.

Selivanov's Theorem

Let k be a finite discrete order.

- (Selivanov 2007) The Wadge degrees of k-valued $\boldsymbol{\Delta}^0_2$ functions $\simeq$ the quotient homomorphic quasiorder on $^{\sqcup}\mathbf{Tree}(k)$.
- (Selivanov 2017) The Wadge degrees of k-valued $\boldsymbol{\Delta}^0_3$ functions $\simeq$ the quotient homomorphic quasiorder on $^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(k))$.

- $\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable trees.
- $^{\sqcup}\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable forests.

Theorem (extending Duparc's and Selivanov's works)

Let Q be a better-quasi-order.

- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_2(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_5(\omega^\omega \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

Hereafter, we will describe a tree and a forest as a term in the following language:

Language $\mathcal{L}(Q)$ for nested Q-labeled trees and forests

1. Constant symbols q (for $q \in Q$).
2. A **2**-ary function symbol $\rightarrow$ (concatenation).
3. An ω-ary function symbol $\sqcup$ (disjoint sum).
4. A unary function symbol $\langle \cdot \rangle$ (labeling).

Example of $\mathcal{L}(2)$-terms

1. The term $0^{\rightarrow}1$ represents open sets (c.e. sets).
2. The term $1^{\rightarrow}0$ represents closed sets (co-c.e. sets).
3. The term $0 \sqcup 1$ represents clopen sets (computable sets).
4. The term $0^{\rightarrow}1^{\rightarrow}0$ represents differences of two open sets (d-c.e. sets).
5. The term $\langle 0^{\rightarrow}1 \rangle$ represents F_σ sets (Σ^0_2 sets).

Definition (the class Σ_T)

For a $\mathcal{L}(\mathcal{Q})$-term T, define the class Σ_T of functions: $\omega^\omega \to \mathcal{Q}$ as follows

1. Σ_q consists only of the constant function $x \mapsto q$.
2. $f \in \Sigma_{\sqcup_i S_i} \iff \exists$ clopen partition $(C_i)_{i\in\omega}$ of ω^ω s.t. $f \upharpoonright C_i$ is in Σ_{S_i}.
3. $f \in \Sigma_{S\to T} \iff$ there is an open set $U \subseteq \omega^\omega$ such that $f \upharpoonright U$ is in Σ_T and $f \upharpoonright (\omega^\omega \setminus U)$ is in Σ_S.
4. $f \in \Sigma_{\langle T\rangle} \iff f = g \circ h$ for some $g \in \Sigma_T$ and $h \in \underset{\sim}{\Sigma}^0_2$.

Note: For the 3rd condition, without affecting its Wadge degree, a function with closed or open domain can be modified to a total function.

1. $\Sigma_{0\to 1} = \underset{\sim}{\Sigma}^0_1$, $\Sigma_{1\to 0} = \underset{\sim}{\Pi}^0_1$, and $\Sigma_{0\sqcup 1} = \underset{\sim}{\Delta}^0_1$.
2. $\Sigma_{0\to 1\to 0}$ = differences of $\underset{\sim}{\Sigma}^0_1$ sets.
3. $\Sigma_{\langle 0\to 1\rangle} = \underset{\sim}{\Sigma}^0_2$, and $\Sigma_{\langle 1\to 0\rangle} = \underset{\sim}{\Pi}^0_2$.
4. No term corresponds to $\underset{\sim}{\Delta}^0_2$ (this reflects the fact that there is no $\underset{\sim}{\Delta}^0_2$-complete set; $\underset{\sim}{\Delta}^0_2$ is divided into unbounded ω_1-many Wadge degrees).

- The classes $\mathbf{\Sigma}_T$'s for terms T in the previous language are enough to exhausts all functions of finite Borel ranks.
- How do we extend this notion to infinite Borel ranks?

- The classes $\mathbf{\Sigma}_T$'s for terms T in the previous language are enough to exhausts all functions of finite Borel ranks.
- How do we extend this notion to infinite Borel ranks?
- It seems natural to define $\mathbf{Tree}^{\xi?}(Q) := \mathbf{Tree}(\bigcup_{\eta<\xi}\mathbf{Tree}^{\eta?}(Q))$.
- It is straightforward to extend the def. of $\mathbf{\Sigma}_T$ to all $T \in \mathbf{Tree}^{\xi?}(Q)$.

- The classes Σ_T's for terms T in the previous language are enough to exhausts all functions of finite Borel ranks.
- How do we extend this notion to infinite Borel ranks?
- It seems natural to define $\mathbf{Tree}^{\xi ?}(Q) := \mathbf{Tree}(\bigcup_{\eta<\xi} \mathbf{Tree}^{\eta ?}(Q))$.
- It is straightforward to extend the def. of Σ_T to all $T \in \mathbf{Tree}^{\xi ?}(Q)$.
- But... for $T \in {}^{\sqcup}\mathbf{Tree}^{\alpha ?}(Q)$, every Σ_T-function is $\underset{\sim}{\Delta}^0_\omega$-measurable!

- The classes Σ_T's for terms T in the previous language are enough to exhausts all functions of finite Borel ranks.
- How do we extend this notion to infinite Borel ranks?
- It seems natural to define $\mathbf{Tree}^{\xi ?}(Q) := \mathbf{Tree}(\bigcup_{\eta<\xi} \mathbf{Tree}^{\eta ?}(Q))$.
- It is straightforward to extend the def. of Σ_T to all $T \in \mathbf{Tree}^{\xi ?}(Q)$.
- But... for $T \in {}^{\sqcup}\mathbf{Tree}^{\alpha ?}(Q)$, every Σ_T-function is $\utilde{\mathbf{\Delta}}^0_\omega$-measurable!
- ...Why?

- The Wadge rank of a $\utilde{\mathbf{\Sigma}}^0_n$-complete set is $\omega_1 \uparrow\uparrow n$.
- The height of the Wadge degrees of Borel sets of finite rank is the first fixed point of the exp. of base ω_1, i.e., $\varepsilon_{\omega_1+1} = \sup_{n<\omega} \omega_1 \uparrow\uparrow n$.
- What is the height of the Wadge degrees of $\utilde{\mathbf{\Delta}}^0_\omega$ sets?
- It is the ω_1-st fixed point of the exp. of base ω_1, i.e., $\varepsilon_{\omega_1+\omega_1}$.
- $(\mathbf{Tree}^{\alpha ?}(2))_{\alpha<\omega_1}$ may only describe the hierarchy between ε_{ω_1+1} and $\varepsilon_{\omega_1+\omega_1}$...?

- ${}^{\sqcup}\mathbf{Tree}^{\omega}(Q)$: the set of all terms in the previous language.
- $\mathbf{Tree}^{\omega}(Q)$: the set of all terms whose outmost symbol is not $\sqcup$.
- Previous Idea: $\mathbf{Tree}(\mathbf{Tree}^{\omega}(Q))$ may describe the rank $\omega + 1$.

- $^{\sqcup}\mathbf{Tree}^{\omega}(Q)$: the set of all terms in the previous language.
- $\mathbf{Tree}^{\omega}(Q)$: the set of all terms whose outmost symbol is not $\sqcup$.
- Previous Idea: $\mathbf{Tree}(\mathbf{Tree}^{\omega}(Q))$ may describe the rank $\omega + 1$.
- Correct Idea: $\mathbf{Tree}^{\omega}(\mathbf{Tree}(Q))$ describes the rank $\omega + 1$.

- $^{\sqcup}\mathbf{Tree}^{\omega}(\mathcal{Q})$: the set of all terms in the previous language.
- $\mathbf{Tree}^{\omega}(\mathcal{Q})$: the set of all terms whose outmost symbol is not $\sqcup$.
- Previous Idea: $\mathbf{Tree}(\mathbf{Tree}^{\omega}(\mathcal{Q}))$ may describe the rank $\omega + 1$.
- Correct Idea: $\mathbf{Tree}^{\omega}(\mathbf{Tree}(\mathcal{Q}))$ describes the rank $\omega + 1$.
- To deal with this, we have to distinguish two labeling functions for inner trees $\mathbf{Tree}(\mathcal{Q})$ and the outer tree $\mathbf{Tree}^{\omega}$.

- Given an ordering $\mathcal{P}$, define the new ordering $\langle\mathcal{P}\rangle^{\omega}$ as follows:
  - $\langle\mathcal{P}\rangle^{\omega} = \{\langle p\rangle^{\omega} : p \in \mathcal{P}\}$.
  - $\langle p\rangle^{\omega} \leq \langle q\rangle^{\omega} \iff p \leq_{\mathcal{P}} q$.
- Obviously $\langle\mathcal{P}\rangle^{\omega}$ is isomorphic to $\mathcal{P}$.
- To avoid a notational confusion, instead of $\mathbf{Tree}^{\omega}(\mathbf{Tree}(\mathcal{Q}))$, we will consider $\mathbf{Tree}^{\omega+1}(\mathcal{Q}) := \mathbf{Tree}^{\omega}(\langle\mathbf{Tree}(\mathcal{Q})\rangle^{\omega})$.
- Continue, e.g. $\mathbf{Tree}^{\omega\cdot 2+6}(\mathcal{Q}) := \mathbf{Tree}^{\omega}(\langle\mathbf{Tree}^{\omega}(\langle\mathbf{Tree}^{6}(\mathcal{Q})\rangle^{\omega})\rangle^{\omega})$, which is isomorphic to $\mathbf{Tree}^{\omega}(\mathbf{Tree}^{\omega}(\mathbf{Tree}^{6}(\mathcal{Q})))$.

We now give the correct description of trees for infinite ranks:

Language $\mathcal{L}_\alpha$ for nested Q-labeled trees and forests

1. Constant symbols q (for $q \in Q$).
2. A $\mathbf{2}$-ary function symbol $\to$ (concatenation).
3. An ω-ary function symbol $\sqcup$ (disjoint sum).
4. A unary function symbol $\langle \cdot \rangle^{\omega^\beta}$ for every $\beta < \alpha$ (ω^β-labeling).

- ${}^{\sqcup}\mathbf{Tree}^{\omega^\alpha}(Q)$: The set of all $\mathcal{L}_\alpha$-terms.
- $\mathbf{Tree}^{\omega^\alpha}(Q)$: The set of all $\mathcal{L}_\alpha$-terms whose outmost symbol is not $\sqcup$.
- Every $\xi < \omega_1$ is uniquely decomposed as $\xi = \omega^\alpha + \beta$.
  Then, inductively define $\mathbf{Tree}^{\xi}(Q) := \mathbf{Tree}^{\omega^\alpha}(\langle \mathbf{Tree}^{\beta}(Q)\rangle^{\omega^\alpha})$.
- $\mathbf{Tree}^{\omega_1}(Q) = \bigcup_{\xi<\omega_1} \mathbf{Tree}^{\xi}(Q)$.
- ${}^{\sqcup}\mathbf{Tree}^{\omega_1}(Q) = \bigcup_{\xi<\omega_1} {}^{\sqcup}\mathbf{Tree}^{\xi}(Q)$.

Definition (the class Σ_T)

For a term T, define the class Σ_T of functions: $\omega^\omega \to \mathcal{Q}$ as follows:

1. Σ_q consists only of the constant function $x \mapsto q$.
2. $f \in \Sigma_{\sqcup_i S_i} \iff \exists$ clopen partition $(C_i)_{i\in\omega}$ of ω^ω s.t. $f \upharpoonright C_i$ is in Σ_{S_i}.
3. $f \in \Sigma_{S\to T} \iff$ there is an open set $U \subseteq \omega^\omega$ such that $f \upharpoonright U$ is in Σ_T and $f \upharpoonright (\omega^\omega \setminus U)$ is in Σ_S.
4. $f \in \Sigma_{\langle T\rangle^{\omega^\alpha}} \iff f = g \circ h$ for some $g \in \Sigma_T$ and $h \in \Sigma^0_{1+\omega^\alpha}$.

A function $f : \omega^\omega \to Q$ is Σ_T-complete if $f \in \Sigma_T$, and every Σ_T-function $g : \omega^\omega \to Q$ is Wadge reducible to f.

Lemma

For any $T \in \mathbf{Tree}^{\omega_1}(Q)$, there is a Σ_T-complete function.

To construct Σ_T-complete functions, we will use complete $\underset{\sim}{\Sigma}^0_\xi$-measurable function on Baire space.

A function $f : \omega^\omega \to Q$ is Σ_T-complete if $f \in \Sigma_T$, and every Σ_T-function $g : \omega^\omega \to Q$ is Wadge reducible to f.

Lemma

For any $T \in \mathbf{Tree}^{\omega_1}(Q)$, there is a Σ_T-complete function.

To construct Σ_T-complete functions, we will use complete $\utilde{\Sigma}^0_\xi$-measurable function on Baire space.

- A Γ-function $g : \omega^\omega \to \omega^\omega$ is complete if for any Γ-function h there is a continuous θ s.t. $h = g \circ \theta$.
- There is no complete total $\utilde{\Sigma}^0_2$-measurable function:$\omega^\omega \to \omega^\omega$. (Otherwise, we would have a greatest $\utilde{\Delta}^0_2$ Wadge degree)
- However, a complete partial $\utilde{\Sigma}^0_2$-measurable function can exist. E.g., the partial limit function $\mathbf{Lim}$ which, given $\langle x(n,s)\rangle_{n,s}$, returns the maximal initial segment of $\langle \lim_s x(n,s)\rangle_n$.
- Similarly, a complete partial $\utilde{\Sigma}^0_\xi$-measurable function exists.

Definition of Σ_T-complete functions

- $\Omega_q : x \mapsto q$, a constant function.
- $\Omega_{\sqcup_i T_i}(n^\frown x) = \Omega_{T_n}(x)$.
- $\Omega_{\langle S\rangle \to F}$ is a function, which given $x \in \omega^\omega$,
  - until seeing the "mind-change" symbol in x, simulate $\Omega_{\langle S\rangle}$.
  - If we see the "mind-change" symbol in x, then erase all previous outputs, and now start to simulate Ω_F.
- $\Omega_{\langle T\rangle^{\omega^\alpha}} = \Omega_T \circ \mathrm{Lim}_{\omega^\alpha}$

where $\mathrm{Lim}_{\omega^\alpha}$ is a complete partial $\underset{\sim}{\Sigma}^0_{1+\omega^\alpha}$-measurable function.

Lemma

Ω_T is Σ_T-complete for any $T \in \mathrm{Tree}^{\omega_1}(Q)$.

We define a quasi-order $\trianglelefteq$ on terms, which is shown to be isomorphic to the Wadge degrees of finite Borel rank.

Definition of $\trianglelefteq$

We inductively define a quasi-order $\trianglelefteq$ on terms as follows:

$$p \trianglelefteq q \iff p \leq_Q q,$$

$$\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff \begin{cases} U \trianglelefteq V & \text{if } \alpha = \beta, \\ \langle U\rangle^{\omega^\alpha} \trianglelefteq V & \text{if } \alpha > \beta, \\ U \trianglelefteq \langle V\rangle^{\omega^\alpha} & \text{if } \alpha < \beta. \end{cases}$$

and if S and T are of the form $\langle U\rangle^{\omega^\alpha \to} \bigsqcup_i S_i$ and $\langle V\rangle^{\omega^\beta \to} \bigsqcup_j T_j$, then

$$S \trianglelefteq T \iff \begin{cases} (\forall i)\ S_i \trianglelefteq T & \text{if } \langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta}, \\ (\exists j)\ S \trianglelefteq T_j & \text{if } \langle U\rangle^{\omega^\alpha} \not\trianglelefteq \langle V\rangle^{\omega^\beta}. \end{cases}$$

Main Lemma 1 ($T \mapsto \Omega_T$ is an embedding)

$S \trianglelefteq T \iff \Omega_S \leq_W \Omega_T$.

Main Lemma 2 ($T \mapsto \Omega_T$ is a surjective)

Every $\underset{\sim}{\mathbf{\Delta}}^0_{1+\xi}$-measurable function $f : \omega^\omega \to \mathcal{Q}$ is Wadge equivalent to Ω_T for some $T \in {}^{\sqcup}\mathbf{Tree}^{\xi}(\mathcal{Q})$.

The key notion in our proofs is the Turing jump operator.

The Turing jump $\boldsymbol{TJ} : \omega^\omega \to \omega^\omega$ has the following properties:

- The Turing jump $\boldsymbol{TJ}$ is an right-complete $\mathbf{\Sigma}^0_2$-computable function: For all $\mathbf{\Sigma}^0_2$-comp. $\boldsymbol{g} : \omega^\omega \to \omega^\omega$, there is a comp. θ s.t. $\boldsymbol{g} = \theta \circ \boldsymbol{TJ}$.
- The Turing jump $\boldsymbol{TJ}$ is an injection whose image is $\mathbf{\Pi}^0_2$.
- The left inverse $\boldsymbol{TJ}^{-1}$ is a computable map with a $\mathbf{\Pi}^0_2$ domain. (For computability, note that $\boldsymbol{X}'$ computes $\boldsymbol{X}$ in a uniform manner!)

The key notion in our proofs is the Turing jump operator.

The Turing jump $\mathbf{TJ} : \omega^\omega \to \omega^\omega$ has the following properties:

- The Turing jump $\mathbf{TJ}$ is an right-complete $\mathbf{\Sigma}^0_2$-computable function: For all $\mathbf{\Sigma}^0_2$-comp. $g : \omega^\omega \to \omega^\omega$, there is a comp. θ s.t. $g = \theta \circ \mathbf{TJ}$.
- The Turing jump $\mathbf{TJ}$ is an injection whose image is $\mathbf{\Pi}^0_2$.
- The left inverse $\mathbf{TJ}^{-1}$ is a computable map with a $\mathbf{\Pi}^0_2$ domain. (For computability, note that X' computes X in a uniform manner!)

Marcone-Montalbán (2011), in "*The Veblen functions for computability theorists*", introduced the notion of the *Turing jump via true stages*.

The MM Turing jump $\mathcal{J}$ is better than the traditional Turing jump $\mathbf{TJ}$:

- The MM Turing jump $\mathcal{J}$ is an right-complete $\mathbf{\Sigma}^0_2$-comp. function.
- The MM Turing jump $\mathcal{J}$ is an injection whose image is closed.
- The left inverse $\mathcal{J}^{-1}$ is a computable map with a closed domain.

$\mathcal{J}^{Z}$: The MM Turing jump relative to Z.

Since the domain $\mathcal{J}^{Z}$ is closed, we can think of it as a total function.

The Jump Inversion Operator $\not\sim$

For a function $f : \omega^\omega \to Q$, define $f^{\not\sim Z} = f \circ (\mathcal{J}^{Z})^{-1}$.

$\mathcal{J}^Z$: The MM Turing jump relative to Z.

Since the domain $\mathcal{J}^Z$ is closed, we can think of it as a total function.

The Jump Inversion Operator $\nrightarrow$

For a function $f : \omega^\omega \to \mathcal{Q}$, define $f^{\nrightarrow Z} = f \circ (\mathcal{J}^Z)^{-1}$.

- $X \leq_T Y \implies f^{\nrightarrow X} \geq_W f^{\nrightarrow Y}$.
- Since the Turing degrees form a upper semilattice, and Wadge degrees of $\mathcal{Q}$-valued func. are well-founded, there must exist Z s.t.

$$\deg_W f^{\nrightarrow Z} = \min_X \deg_W f^{\nrightarrow X}.$$

- Define $f^{\nrightarrow} = f^{\nrightarrow Z}$ for such a Z.

Key Lemma (Jump Inversion)

$(h \circ \mathbf{Lim})^{\not\to} \equiv_W h$. In particular, $\Omega^{\not\to}_{\langle T \rangle} \equiv_W \Omega_T$.

- Our jump inversion theorem follows from (the uniform version of) the Friedberg jump inversion theorem.
- Our above result seems related to Brattka's work on "*A Galois connection between Turing jumps and limits*."

Key Lemma (Jump Inversion)

$(h \circ \mathbf{Lim})^{\curvearrowright} \equiv_W h$. In particular, $\Omega^{\curvearrowright}_{\langle T \rangle} \equiv_W \Omega_T$.

- Our jump inversion theorem follows from (the uniform version of) the Friedberg jump inversion theorem.
- Our above result seems related to Brattka's work on "*A Galois connection between Turing jumps and limits*."

Definition (Louveau and Saint-Raymond 1990)

A function $f : \omega^\omega \to \mathcal{Q}$ is *self-dual* if

$$\exists \text{ continuous } \theta \text{ s.t. for all } x \in \omega^\omega,\ f \circ \theta(x) \not\leq_{\mathcal{Q}} f(x).$$

Key Lemma (Jump-Inversion-Cancellation)

Assume $f^{\curvearrowright}$ or $g^{\curvearrowright}$ is non-self-dual. Then $f^{\curvearrowright} \leq_W g^{\curvearrowright} \implies f \leq_W g$.

- This follows from the (Borel) determinacy.
- The argument is almost the same as Duparc (2001), "*Wadge hierarchy and Veblen hierarchy Part I: Borel sets of finite rank*."

$\mathcal{J}^{\omega^\alpha, Z}$: The ω^α-th MM Turing jump operator relative to Z.
The left-inverse $(\mathcal{J}^{\omega^\alpha, Z})^{-1}$ is also continuous with closed domain.
Define $f^{\nrightarrow\omega^\alpha}$ as one realizing the least Wadge deg. among $f \circ (\mathcal{J}^{\omega^\alpha, Z})^{-1}$.

Transfinite Jump Inversion Theorem

$(h \circ \mathbf{Lim}_{\omega^\alpha})^{\nrightarrow\omega^\alpha} \equiv_W h$. In particular, $\Omega^{\nrightarrow\omega^\alpha}_{\langle T\rangle^{\omega^\alpha}} \equiv_W \Omega_T$.

- This follows from (the uniform version of) the transfinite jump inversion theorem.

Transfinite Jump-Inversion-Cancellation

Assume $f^{\nrightarrow\omega^\alpha}$ or $g^{\nrightarrow\omega^\alpha}$ is non-self-dual. Then $f^{\nrightarrow\omega^\alpha} \leq_W g^{\nrightarrow\omega^\alpha} \implies f \leq_W g$.

- This again follows from the (Borel) determinacy.

A function $f : \omega^\omega \to Q$ is α-*stable* if

- f is initializable, i.e., $(\exists g \equiv_W f)(\forall \sigma \in \omega^{<\omega})\ g \upharpoonright [\sigma] \equiv_W g$.
- $f^{\not\to\omega^\beta} \equiv_W f$ for any $\beta < \alpha$.

Key Technical Lemma on α-Stability

1. $\Omega_{\langle T\rangle^{\omega^\alpha}}$ is α-stable.
2. If f is α-stable, then $f^{\not\to\omega^\alpha}$ is non-self-dual.

- (1) again uses the (transfinite) jump inversion theorem.
- (2) for $\alpha = 0$ just follows from a simple finite extension method as in Duparc (2001).
- (2) for $\alpha > 0$ involves some inverse limit construction.

Main Lemma 1

$S \trianglelefteq T \iff \Omega_S \leq_W \Omega_T$.

Main Lemma 2

Every $\underset{\sim}{\mathbf{\Delta}}^0_{1+\xi}$-measurable function $f : \omega^\omega \to \mathcal{Q}$ is Wadge equivalent to Ω_T for some $T \in {}^{\sqcup}\mathbf{Tree}^{\xi}(\mathcal{Q})$.

Proof of Main Lemma 1 ($S \trianglelefteq T \iff \Omega_S \leq_W \Omega_T$):

- If $T \in {}^{\sqcup}\mathbf{Tree}(Q)$ is constructed from $\sqcup, \rightarrow$, the proof is the same as the classical $\underset{\sim}{\mathbf{\Delta}}^0_2$ case.
- We only show that $\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff \Omega_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega_{\langle V\rangle^{\omega^\beta}}$.
- If $\alpha = \beta$, by definition and Induction Hypothesis: $\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff U \trianglelefteq V \iff \Omega_U \leq_W \Omega_V$.
- By the jump inversion: $\Omega_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega_{\langle V\rangle^{\omega^\alpha}} \implies \Omega_U \equiv_W \Omega^{\not\,\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega^{\not\,\omega^\alpha}_{\langle V\rangle^{\omega^\alpha}} \equiv_W \Omega_V$.
- Since $\Omega_{\langle U\rangle^{\omega^\alpha}}$ is α-stable, $\Omega^{\not\,\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}}$ is non-self-dual.

  Thus, by the jump inversion and the jump-inversion-cancellation, $\Omega_U \leq_W \Omega_V \implies \Omega^{\not\,\omega^\alpha}_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega^{\not\,\omega^\alpha}_{\langle V\rangle^{\omega^\alpha}} \implies \Omega_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega_{\langle V\rangle^{\omega^\alpha}}$.
- If $\alpha > \beta$, by definition and Induction Hypothesis: $\langle U\rangle^{\omega^\alpha} \trianglelefteq \langle V\rangle^{\omega^\beta} \iff \langle U\rangle^{\omega^\alpha} \trianglelefteq V \iff \Omega_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega_V$.
- By the α-stability of $\Omega_{\langle U\rangle^{\omega^\alpha}}$ and by the jump inversion: $\Omega_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega_{\langle V\rangle^{\omega^\alpha}} \implies \Omega_{\langle U\rangle^{\omega^\alpha}} \equiv_W \Omega^{\not\,\omega^\beta}_{\langle U\rangle^{\omega^\alpha}} \leq_W \Omega^{\not\,\omega^\beta}_{\langle V\rangle^{\omega^\beta}} \equiv_W \Omega_V$.
- The converse direction is the same as before.

Proof of Main Lemma 2 ($(\forall f \text{ Borel})(\exists T)\ f \equiv_W \Omega_T$):

- If f is self-dual, then it follows from the standard argument.
- If f is non-self-dual, and not $\mathbf{0}$-stable, then consider $F(f) = \{X \in \omega^\omega : (\exists \sigma \prec X)\ f \upharpoonright [\sigma] \equiv_W f\}$, and:
  - The "quotient" Q_f is defined by the initializable part $f \upharpoonright F(f)$.
  - The "reminder" R_f is defined by $f \upharpoonright (\omega^\omega \setminus F(f))$.

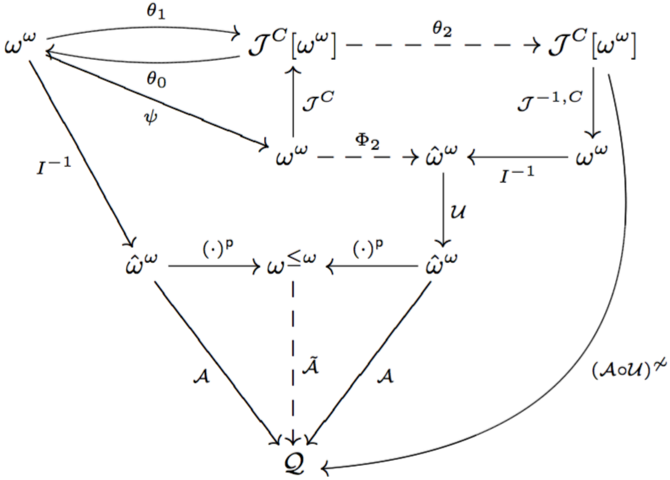
  Then $Q_f, R_f <_W f$ and $f \equiv_W Q_f{}^\to R_f$.
  Thus, by Induction Hypothesis, $f \equiv_W \Omega_S{}^\to \Omega_T$ for some S and T.

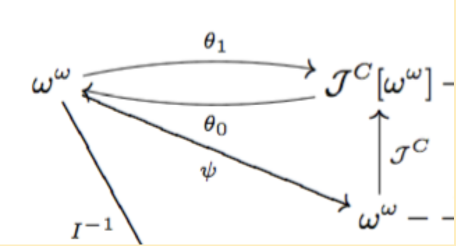
- Assume that f is α-stable, but not $(\alpha + 1)$-stable.
- Since f is not $(\alpha + 1)$-stable, we have $f^{\nleftarrow \omega^\alpha} <_W f$.
- Since f is α-stable, $f^{\nleftarrow \omega^\alpha}$ is non-self-dual.
- By Induction Hypothesis, $f^{\nleftarrow \omega^\alpha} \equiv_W \Omega_T$ for some tree T.
- By the jump-inversion, we get $f^{\nleftarrow \omega^\alpha} \equiv_W \Omega_T \equiv_W \Omega^{\nleftarrow \omega^\alpha}_{\langle T \rangle^{\omega^\alpha}}$.
- By the jump-inversion-cancellation, we conclude $f \equiv_W \Omega_{\langle T \rangle^{\omega^\alpha}}$.

Four key tools:

- (Jump inversion) $\Omega^{\not\to\omega^\alpha}_{\langle T\rangle^{\omega^\alpha}} \equiv_W \Omega_T$.
- (Jump-inversion-cancellation) Assume $f^{\not\to\omega^\alpha}$ or $g^{\not\to\omega^\alpha}$ is non-self-dual. Then $f^{\not\to\omega^\alpha} \leq_W g^{\not\to\omega^\alpha} \implies f \leq_W g$.
- (Stability) $\Omega_{\langle T\rangle^{\omega^\alpha}}$ is α-stable.
- (Stable-to-non-self-dual) If f is α-stable, then $f^{\not\to\omega^\alpha}$ is non-self-dual.

Proof of The Jump Inversion Lemma $\Omega^{\nearrow}_{\langle T \rangle} \equiv_W \Omega_T$



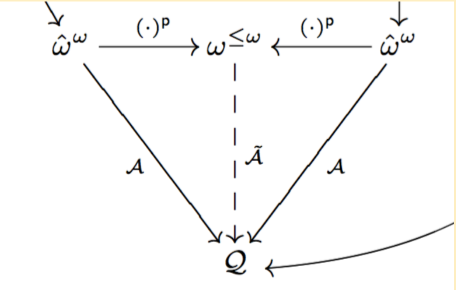


- (Jump Inversion Theorem) $(\forall C, X)(\exists Y)\ (Y \oplus C)' \equiv_T X \oplus C'$.
- $X \mapsto Y$ is uniform, that is, $\exists C'$-computable ψ s.t.

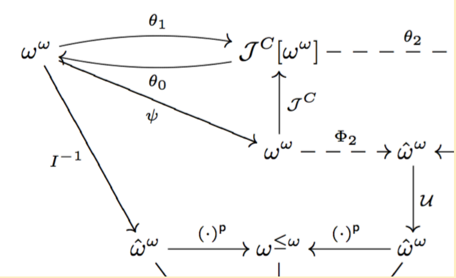
$$(\forall X, C) \quad (\psi(X) \oplus C)' \equiv_T X \oplus C'.$$

- The Turing equivalence is also uniform:
  - $\geq_T$ is witnessed by a computable $\theta_0 : (\psi(X) \oplus C)' \mapsto X$.
  - $\leq_T$ is witnessed by a C'-computable $\theta_1 : X \mapsto (\psi(X) \oplus C)'$.
- That is, $\theta_0 \circ \mathcal{J}^C \circ \psi = \mathbf{id}$ and $\theta_1 = \mathcal{J}^C \circ \psi$.

Proof of The Jump Inversion Lemma $\Omega^{\nrightarrow}_{\langle T\rangle} \equiv_W \Omega_T$

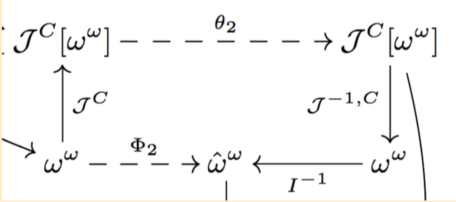


- $\hat{\omega} = \omega \cup \{\mathbf{pass}\}$; $(X)^{\mathbf{p}}$ is the result of removing all **pass**'es from X.
- (Duparc) $\mathcal{A}$ is *conciliatory* if $\exists\ \tilde{\mathcal{A}}$ s.t. $\tilde{\mathcal{A}}((X)^{\mathbf{p}}) = \mathcal{A}(X)$ for all X.
- $\mathcal{A} := \Omega_T$ is conciliatory for any $T \in \mathbf{Tree}^{\omega_1}(\mathcal{Q})$.
- Let $I : \hat{\omega}^\omega \to \omega^\omega$ is a computable homeomorphism obtained from a computable bijection between $\hat{\omega}$ and ω.



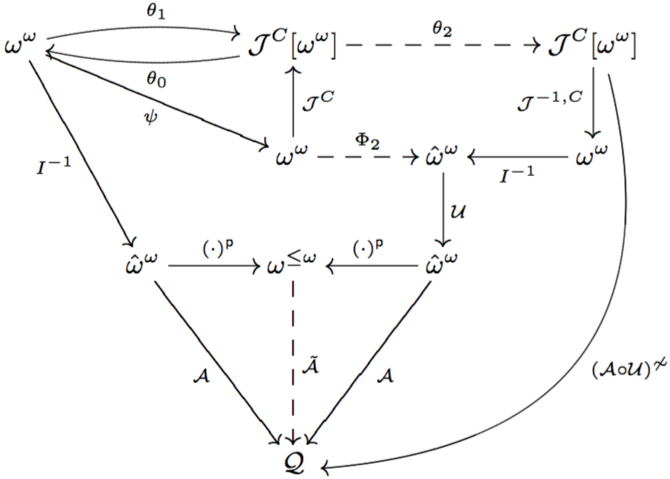
- $\mathcal{U} := \mathbf{Lim}$ is a complete conciliatory $\underset{\sim}{\Sigma}^0_2$-measurable function.
- (Completeness) $\forall f \in \underset{\sim}{\Sigma}^0_2 \exists$ cont. $\Phi \forall X\ (f(X))^{\mathsf{p}} = (\mathcal{U} \circ \Phi(X))^{\mathsf{p}}$.
- Indeed, since $I^{-1} \circ \theta_0 \circ \mathcal{J}^C$ is $\Sigma^{0,C}_2$-computable, there is a C-computable Φ_2 s.t. $(I^{-1} \circ \theta_0 \circ \mathcal{J}^C(X))^{\mathsf{p}} = (\mathcal{U} \circ \Phi_2(X))^{\mathsf{p}}$.
- That is, $(\cdot)^{\mathsf{p}} \circ I^{-1} \circ \theta_0 \circ \mathcal{J}^C = (\cdot)^{\mathsf{p}} \circ \mathcal{U} \circ \Phi_2(X))$.

Proof of The Jump Inversion Lemma $\Omega^{\nsim}_{\langle T\rangle} \equiv_W \Omega_T$



- Recall: $\mathcal{J}^C$ is an right-complete $\Sigma^{0,C}_2$-computable function:
  For all $\Sigma^{0,C}_2$-comp. $g : \omega^\omega \to \omega^\omega$, $\exists$ computable θ s.t. $g = \theta \circ \mathcal{J}^C$.
- $\mathcal{J}^C \circ I \circ \Phi_2 : \omega^\omega \to \omega^\omega$ is $\Sigma^{0,C}_2$-computable.
- Thus, there is a computable θ_2 s.t. $\mathcal{J}^C \circ I \circ \Phi_2 = \theta_2 \circ \mathcal{J}^C$.

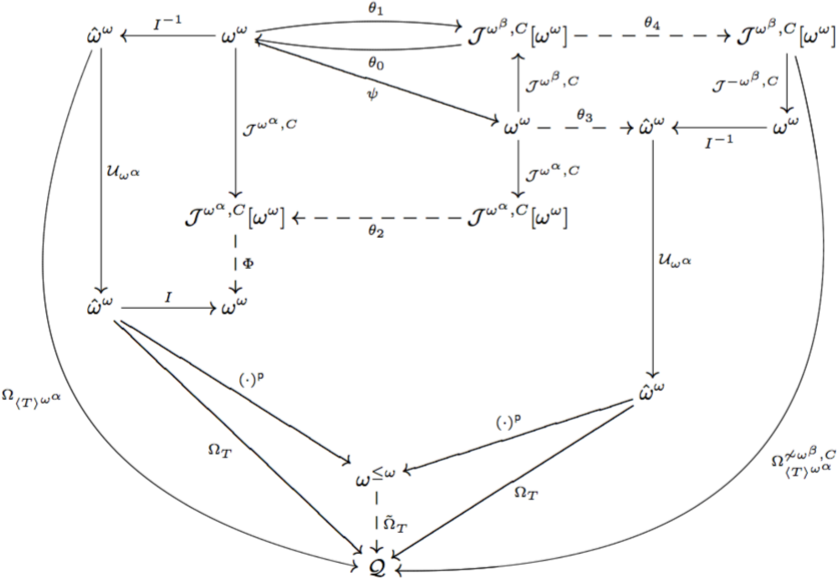
Proof of The Jump Inversion Lemma $\Omega^{\nsim}_{\langle T\rangle} \equiv_W \Omega_T$



The previous argument shows that the above diagram is commutative:

$$\Omega_T = \mathcal{A} \leq_W (\mathcal{A} \circ \mathcal{U})^{\nsim} = \Omega^{\nsim}_{\langle T\rangle} \text{ via } \theta_2 \circ \theta_1 \circ I.$$

Proof of The α-Stability Lemma



Proof of the "stable-to-non-self-dual" lemma is more complicated.

Main Theorem

Let Q be BQO. The following structures are isomorphic:

1. The Wadge degrees of Q-valued $\mathbf{\Delta}^0_{1+\xi}$-measurable functions.
2. The many-one-on-a-cone degrees of Q-valued $(\equiv_T, \equiv_m)$-degree-invariant $\mathbf{\Delta}^0_{1+\xi}$-functions.
3. The quotient order of $(\mathrm{Tree}^{\xi}(Q), \trianglelefteq)$.

T. Kihara and A. Montalbán,
The uniform Martin's conjecture for many-one degrees,
to appear in Transactions of the American Mathematical Society.

T. Kihara and A. Montalbán,
On the structure of the Wadge degrees of BQO-valued Borel functions,
to appear in Transactions of the American Mathematical Society.