Linearisation in model theory
(an ideological address)

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In this talk

The talk in a nutshell

A is an abelian group and $R \leq \mathrm{End}_{\mathrm{def}}(A)$ a ring of def. endomorphisms.
Under assumptions:

- on the algebraic behaviour of the action of R on A (usual Schur stuff);
- on the logical behaviour of R and A (sufficient definability);
- on the logical theory of the whole ("finite-dimensionality"),

then A is actually a vector space, and R acts by scalars.

We care because:

- intrinsic beauty;
- a field, with coordinates, is easier to study than an abstract structure;
- extends work by: Schur, Artin, Zilber, Poizat, Wagner. . . ;
- it finally puts them in the proper setting.

Let's go!

Model-theoretic algebra; degrees of definability

- In model-theoretic algebra, structures are given to us which satisfy some logical constraints, and we aim at identifying them.
- The logical constraints are often formulated on the definable class.
  In this talk, definable always means *interpretable with parameters*.
- Typical assumption: on the definable class, there is a "dimension".
  Eg.: Morley rank, *o*-minimal dimension, ... (def. comes later ↗)
- Today we need a bit more than definability.
  A set is *invariant* if it is a bounded union of type-definable sets.
  (Afraid of invariance? In practice, $\bigvee$-definability suffices = countable union of definable sets.)

And we begin with a general question.

The origin of fields

Question

Where do fields come from (in model theory)?

1. Hilbert-Desargues: an arguesian projective plane defines a skew-field.
   Fascinating, very useful even in group theory!
   Eg. (Nesin): a bad group has no involutions.
2. Heisenberg-Malcev: a nice nilpotent group defines a ring.
   The nicer the group, the more field-like the ring.
   To my knowledge, never used in groups of finite Morley rank!
3. And of course, there is the Artin-Schur-Zilber thing. . .

Schur's Lemma

Theorem (Schur's Lemma)

Let A be an abelian group and $R \leq \mathrm{End}(A)$ be a ring acting on it. Suppose that A is simple as an R-module. Then the centraliser/covariance ring

$$C := \mathrm{Cov}(R) := \{\lambda \in \mathrm{End}(A) : \forall r \in R\ \lambda \circ r = r \circ \lambda\}$$

is a skew-field over which A is a vector space. R is linear.

Proof.

Let $\lambda \in C \setminus \{0\}$.

- $\ker \lambda$ is R-invariant, so by simplicity $\ker \lambda = \{0\}$ or A; A is out.
- Likewise $\mathrm{im}\, \lambda = \{0\}$ or A and $\{0\}$ is out.
- Then clearly $\lambda^{-1} \in C$. □

— The whole point is to find a *definable* version, i.e. to make C definable.

Zilber's version from the 80's

First we relax simplicity to the definable category:

Definition (please remember this one)

The definable, abelian group A is *X-minimal* if it has no definable, infinite, proper, X-invariant subgroup.

Theorem ("Zilber's Field Theorem")

Let $S = A \rtimes H$ be an abelian-by-abelian, connected group of finite Morley rank with A H-minimal.Then S defines an infinite field.

Problem: group-theorists tend to neglect rings.
Zilber's Field Theorem should actually be something like:

Theorem (Schur-Artin-Zilber linearisation theorem)

In a theory of finite Morley rank, if A is a definable, abelian group and $R \leq \mathrm{End}_{\mathrm{def}}(A)$ is a $\vee$-definable, commutative ring such that A is R-minimal, then $\mathrm{Cov}(R)$ is a definable field.

Similar results

Theorem (Loveys-Wagner)

In a theory of finite Morley rank, if A is a definable, abelian, torsion-free group and $R \leq \mathrm{End}_{\mathrm{def}}(A)$ is a $\vee$-definable ring such that A is R-minimal, then $\mathrm{Cov}(R)$ is a definable field.

Theorem (folklore; perhaps not even written)

In an o-minimal theory, if A is a definable, abelian group and $R \leq \mathrm{End}_{\mathrm{def}}(A)$ is a $\vee$-definable ring such that A is R-minimal, then $\mathrm{Cov}(R)$ is a definable field.

And at least three more *which all require(d) distinct proofs*.
In my opinion none was really well-phrased as they forgot Emil Artin's fundamental contribution. We need sophistication.

Looking for the conclusion

So R acts on A, and we are looking for the theorem.
Reverse engineering: if R acts by scalars, say $R \leq \mathbb{K}$, then $C := \mathrm{Cov}(R) \geq \mathrm{Cov}(\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(A)$.
One expects equality, and then $\mathbb{K} = \mathrm{Cov}(C) = \mathrm{Cov}(\mathrm{Cov}(R))$.
(It is well-known to algebraists that a *double centraliser mimicks closure*!)
So our desired statement will take the form:

Theorem

...
Then $\mathbb{K} = \mathrm{Cov}(C)$ is a definable skew-field, A is a finite-dimensional vector space over $\mathbb{K}$, and $R \leq \mathbb{K}$ acts by scalars and $C = \mathrm{End}_{\mathbb{K}}(A)$. If R is commutative then so is $\mathbb{K}$.

(We have *not* explained definability of $\mathbb{K}$, since a double centraliser inside $\mathrm{End}_{\mathrm{def}}(A)$ need not be definable.)

Looking for the algebraic assumptions

Zilber assumed minimality of A as an R-module, but this is no longer what we want: if $R \leq \mathbb{K}$ then A is certainly *not* R-minimal.

The algebraic assumption will be:

Theorem

...

Suppose that:

- $C := \mathrm{Cov}(R)$ is unbounded (←contains a "large" type-def. set);
- A is C-minimal.

Then $\mathbb{K} = \mathrm{Cov}(C)$ is a definable skew-field, A is a finite-dimensional vector space over $\mathbb{K}$, and $R \leq \mathbb{K}$ acts by scalars and $C = \mathrm{End}_{\mathbb{K}}(A)$. If R is commutative then so is $\mathbb{K}$.

Looking for the logical assumptions

We must handle simultaneously the finite MR and *o*-minimal cases. Common feature: there is a nice dimension function. (This is where I'm taking you next.) The final result will be:

Theorem ("R-C linearisation theorem")

In a finite-dimensional theory, let A be a definable, connected, abelian group. Let $R \leq \mathrm{End}_{\mathrm{def}}(A)$ be an invariant subring; let $C = \mathrm{Cov}(R) = \{c \in \mathrm{End}_{\mathrm{def}}(A) : \forall r \in R\ cr = rc\}$ be its centraliser. Suppose that:

- *C is unbounded;*
- *A is C-minimal.*

Then $\mathbb{K} = \mathrm{Cov}(C)$ is a definable skew-field, A is a finite-dimensional vector space over $\mathbb{K}$, and $R \leq \mathbb{K}$ acts by scalars and $C = \mathrm{End}_{\mathbb{K}}(A)$. If R is commutative then so is $\mathbb{K}$.

The Definition

Definition

A theory T is *finite-dimensional* if there is an integer-valued dimension function dim on definable subsets of models of T such that:

1. $\dim(X) = 0$ if and only if X is finite;
2. dim is automorphism-invariant: $\dim(\pi(x, a))$ only depends on $\operatorname{tp}(a)$;
3. dim is (weakly) increasing: if $X \subseteq Y$ then $\dim(X) \leq \dim(Y)$;
4. dim is additive: if $f : X \to Y$ is a definable map whose fibres all have constant dimension n, then $\dim(X) = n + \dim(Y)$.

This covers finite Morley rank and *o*-minimal dimension (actually more).

Tools and non-tools

DON'Ts:

- Forget about the DCC.
  (Although DCC holds in fMR and *o*-minimal, not true here.)
- Likewise, no connected components.
- Forget about "Macintyre-style" classification results definable fields.
  (So we'll have little information on the algebraic properties of $\mathbb{K}$.)
- No Chevalley-Zilber generation lemma (aka "Indecomposability theorem") either — interestingly, we don't care.

DO's:

- dim-connected groups: on which we salvage a DCC and ACC.
- Some control on uniform families of field automorphisms.

The setting

Proof —

I would like to sketch the main ideas for a couple of slides. From now on:

- A is a definable, abelian, absolutely connected group,
- $R \leq \mathrm{End}_{\mathrm{def}}(A)$ is an invariant ring,
- $C = \mathrm{Cov}(R) = \{c \in \mathrm{End}_{\mathrm{def}}(A) : \forall r \in R cr = rc\}$ is unbounded,
- A is C-minimal.

We are trying to linearise.

Key idea #1: Lines

To linearise is to find *lines*.

- Attempt #1: definable R-submodules minimal as such.
  Problem: no DCC
- Attempt #2: definable, dim-conn. R-submodules minimal as such.
  Problem: no "complete reducibility" (see below).
- Key idea: in lin. alg., lines are both *subobjects* and *quotient objects*.
- Attempt #3 (works): a line is a $cA \leq A$ (for $c \in C$) of minimal dim.
  Then one can prove:

  each line L has "quasi-complements" $H \leq A$ with $A = L \oplus H$ (quasi-direct sum, viz. finite intersection allowed).

Key idea #2: Grouping lines

A priori R could act by *different* scalars on various pieces of A.
So we must find an eq. relation $\sim$ on lines s.t. $L_1 \sim L_2$ iff at the end of the day, R induces the same scalar action on L_1 and L_2.

- Attempt #1 (historical): $L_1 \sim L_2$ iff $\mathrm{Ann}_R(L_1) = \mathrm{Ann}_R(L_2)$
  Problem: simply too coarse (technical counter-example)!
- Attempt #2: $L_1 \sim L_2$ iff there is $c \in C$ with $cL_1 = L_2$.
  Problems: lack of global maps + what about finite kernels?
- Attempt #3 (works): $L_1 \sim L_2$ iff there are finite $F_i \leq L_i$ and a definable, R-covariant $f : L_1/F_1 \simeq L_2/F_2$.
  Then one can prove: $\{\text{lines}\}/\sim$ is finite.
  As a matter of fact using model theory + finiteness, one *finally* proves that *there are no finite R-submodules*.

Key idea #3: Finding the fields

Here we use a key lemma.

Lemma (nothing to do with Chevalley-Zilber "indecomposability")

In a finite-dimensional theory, let L be a definable abelian group. Suppose that there is an invariant, unbounded domain $R \leq \mathrm{End}_{\mathrm{def}}(L)$ acting by automorphisms. Then the skew-field of fractions $\mathbb{K}$ of R exists and is definable; L is definably a finite-dimensional $\mathbb{K}$-vector space.

Now fix a line L.

- If R is unbounded, we are done.
- If R is bounded, recall that there are only finitely many $\sim$-classes. So "unboundedly often" $cL \sim L$. But there are no finite R-modules, so we may reverse the arrows.
  So: $\{c \in C \text{ fixing } L\}$ is unbounded. Apply lemma to this one instead!

Key idea #4: Tidying things up

We just follow Artin.

- At this stage, each line L is a v.s. over some definable field $\mathbb{K}_L$.
- For φ a $\sim$-class, let $A_\varphi = \sum_{L \in \varphi} L$.
  Then let $C_\varphi = \{c_{|A_\varphi} : c \in C\}$ and $\mathbb{K}_\varphi = \mathrm{Cov}(C_\varphi) \leq \mathrm{End}_{\mathrm{def}}(A_\varphi)$ (a double centraliser, as always).
  It is not hard to see that $\mathbb{K}_\varphi$ is a definable field.
- And it is a matter of understanding what you have been doing so far to realise that *everything is now trivial*.
- *A posteriori*, all lines are $\sim$-equivalent! $\square$

The real thing

As a matter of fact we proved something *much more general*.

Theorem ("R-C-N linearisation theorem")

In a finite-dimensional theory, let A be a definable, connected, abelian group. Let $R \leq \mathrm{End}_{\mathrm{def}}(A)$ be an invariant subring; let $C = \{c \in \mathrm{End}_{\mathrm{def}}(A) : \forall r \in R\ cr = rc\}$ be its centraliser and $N = \{n \in \mathrm{End}_{\mathrm{def}}(A) : nR = Rn\}$ be its normaliser. Suppose that:

- *C is infinite;*
- *N is unbounded;*
- *A is N-minimal.*

Then there is a canonical, finite family of infinite definable, pairwise definably isomorphic skew-fields $(\mathbb{K}_\varphi)_{\varphi \in \Phi}$ over which is definably a piecewise finite-dimensional vector space. Moreover R acts by piecewise scalars and N by piecewise semi-linear maps. C acts on each A_φ as $\mathrm{End}_{\mathbb{K}_\varphi}(A_\varphi)$. If R is commutative so are the fields.

The talk in a nutshell, da capo

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then A was actually a vector space, and R acted by scalars.

These finite-dimensional theories beg to be studied further.

Thank you for your attention!