Pseudofinite groups and tame arithmetic regularity

Gabriel Conant
Notre Dame

23 July 2018
Logic Colloquium
University of Udine

Regularity

Szemerédi's Regularity Lemma (1976): Any sufficiently large finite graph can be nontrivially partitioned into a small number of pieces so that most pairs of pieces have regular edge distribution.

Regularity

Szemerédi's Regularity Lemma (1976): Any sufficiently large finite graph can be nontrivially partitioned into a small number of pieces so that most pairs of pieces have regular edge distribution.

Green (2005): For any $A \subseteq (\mathbb{Z}/p\mathbb{Z})^n$, there is a subgroup H of small index such that A is uniformly distributed inside almost all cosets of H.

Regularity

Szemerédi's Regularity Lemma (1976): Any sufficiently large finite graph can be nontrivially partitioned into a small number of pieces so that most pairs of pieces have regular edge distribution.

Green (2005): For any $A \subseteq (\mathbb{Z}/p\mathbb{Z})^n$, there is a subgroup H of small index such that A is uniformly distributed inside almost all cosets of H.

This is a special case of Green's "arithmetic regularity lemma for finite abelian groups".

Regularity

Szemerédi's Regularity Lemma (1976): Any sufficiently large finite graph can be nontrivially partitioned into a small number of pieces so that most pairs of pieces have regular edge distribution.

Green (2005): For any $A \subseteq (\mathbb{Z}/p\mathbb{Z})^n$, there is a subgroup H of small index such that A is uniformly distributed inside almost all cosets of H.

This is a special case of Green's "arithmetic regularity lemma for finite abelian groups".

$\longleftarrow$ highly structured   highly random $\longrightarrow$

😁 😐 ☹️ 😐 😁

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)
- Alon, Fox, Zhao (15 Jan 2018) VC-sets in finite abelian groups of bounded exponent (quantitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)
- Alon, Fox, Zhao (15 Jan 2018) VC-sets in finite abelian groups of bounded exponent (quantitative)
- Sisask (8 Feb 2018) VC-sets in finite abelian groups (quantitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)
- Alon, Fox, Zhao (15 Jan 2018) VC-sets in finite abelian groups of bounded exponent (quantitative)
- Sisask (8 Feb 2018) VC-sets in finite abelian groups (quantitative)
- C., Pillay, Terry (12 Feb 2018) VC-sets in finite groups, and in finite groups of bounded exponent (qualitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)
- Alon, Fox, Zhao (15 Jan 2018) VC-sets in finite abelian groups of bounded exponent (quantitative)
- Sisask (8 Feb 2018) VC-sets in finite abelian groups (quantitative)
- C., Pillay, Terry (12 Feb 2018) VC-sets in finite groups, and in finite groups of bounded exponent (qualitative)
- Terry, Wolf (17 May 2018) stable sets in finite abelian groups (quantitative)

Tame arithmetic regularity

- Terry, Wolf (5 Oct 2017) stable sets in $(\mathbb{Z}/p\mathbb{Z})^n$ (quantitative)
- C., Pillay, Terry (17 Oct 2017) stable sets in finite groups (qualitative)
- Alon, Fox, Zhao (15 Jan 2018) VC-sets in finite abelian groups of bounded exponent (quantitative)
- Sisask (8 Feb 2018) VC-sets in finite abelian groups (quantitative)
- C., Pillay, Terry (12 Feb 2018) VC-sets in finite groups, and in finite groups of bounded exponent (qualitative)
- Terry, Wolf (17 May 2018) stable sets in finite abelian groups (quantitative)
- C. (15 June 2018) VC-sets in finite groups of bounded exponent (99% quantitative), plus some further results

VC-sets in groups

Given a group G and a subset $A \subseteq G$, let $\mathrm{VC}(A)$ denote the VC-dimension of $\{gA : g \in G\}$.

In other words, $\mathrm{VC}(A) \geq d$ if and only if there is $X \subseteq G$ such that $|X| = d$ and $\{X \cap gA : g \in G\} = \mathcal{P}(X)$.

VC-sets in groups

Given a group G and a subset $A \subseteq G$, let $\mathrm{VC}(A)$ denote the VC-dimension of $\{gA : g \in G\}$.

In other words, $\mathrm{VC}(A) \geq d$ if and only if there is $X \subseteq G$ such that $|X| = d$ and $\{X \cap gA : g \in G\} = \mathcal{P}(X)$.

Remark: $\mathrm{VC}(A)$ is finite if and only if the formula $xy \in A$ is NIP in the structure $(G, \cdot, A)$.

VC-sets in finite abelian groups of bounded exponent

Theorem (Alon, Fox, Zhao 2018)

Suppose G is a finite abelian group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$.

VC-sets in finite abelian groups of bounded exponent

Theorem (Alon, Fox, Zhao 2018)

Suppose G is a finite abelian group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. *Then, for any $\epsilon > 0$, there are:*

* *a subgroup $H \leq G$, of index $O_{r,d}((1/\epsilon)^{d+1})$, and*
* *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon |G|$.

VC-sets in finite abelian groups of bounded exponent

Theorem (Alon, Fox, Zhao 2018)

Suppose G is a finite abelian group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a subgroup $H \leq G$, of index $O_{r,d}((1/\epsilon)^{d+1})$, and*
- *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon |G|$.

Given a finite group G, a subset $A \subseteq G$, and $\epsilon > 0$, define

$$\mathrm{Stab}_\epsilon(A) = \{x \in G : |xA \triangle A| \leq \epsilon |G|\}.$$

VC-sets in finite abelian groups of bounded exponent

Theorem (Alon, Fox, Zhao 2018)

Suppose G is a finite abelian group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a subgroup $H \leq G$, of index $O_{r,d}((1/\epsilon)^{d+1})$, and*
- *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon|G|$.

Given a finite group G, a subset $A \subseteq G$, and $\epsilon > 0$, define

$$\mathrm{Stab}_\epsilon(A) = \{x \in G : |xA \triangle A| \leq \epsilon|G|\}.$$

Idea: In abelian groups of bounded exponent, stabilizers of VC-sets contain large subgroups.

Ingredients of the proof

Let G be a finite group.

Ingredients of the proof

Let G be a finite group.

Corollary of Haussler's Packing Lemma

If $A \subseteq G$ and $\mathrm{VC}(A) \leq d$, then $|\operatorname{Stab}_\epsilon(A)| \geq (\epsilon/30)^d |G|$.

Ingredients of the proof

Let G be a finite group.

Corollary of Haussler's Packing Lemma

If $A \subseteq G$ and $\mathrm{VC}(A) \leq d$, then $|\,\mathrm{Stab}_\epsilon(A)| \geq (\epsilon/30)^d |G|$.

Bogolyubov-Ruzsa Lemma (bounded exponent case)

Assume G is abelian of exponent r. Fix a nonempty set $S \subseteq G$, with $|S + S| \leq k|S|$. Then $2S - 2S$ contains a subgroup H of size $\Omega_{r,k}(|S|)$.

Ingredients of the proof

Let G be a finite group.

Corollary of Haussler's Packing Lemma

If $A \subseteq G$ and $\mathrm{VC}(A) \leq d$, then $|\,\mathrm{Stab}_\epsilon(A)| \geq (\epsilon/30)^d |G|$.

Bogolyubov-Ruzsa Lemma (bounded exponent case)

Assume G is abelian of exponent r. Fix a nonempty set $S \subseteq G$, with $|S+S| \leq k|S|$. Then $2S - 2S$ contains a subgroup H of size $\Omega_{r,k}(|S|)$.

Note: If $S \subseteq G$ and $|S| \geq \alpha|G|$, then $|S+S| \leq |G| \leq \alpha^{-1}|S|$.

Ingredients of the proof

Let G be a finite group.

Corollary of Haussler's Packing Lemma

If $A \subseteq G$ and $\mathrm{VC}(A) \leq d$, then $|\mathrm{Stab}_\epsilon(A)| \geq (\epsilon/30)^d |G|$.

Bogolyubov-Ruzsa Lemma (bounded exponent case)

Assume G is abelian of exponent r. Fix a nonempty set $S \subseteq G$, with $|S+S| \leq k|S|$. Then $2S - 2S$ contains a subgroup H of size $\Omega_{r,k}(|S|)$.

Note: If $S \subseteq G$ and $|S| \geq \alpha|G|$, then $|S+S| \leq |G| \leq \alpha^{-1}|S|$.

Lemma (Alon, Fox, Zhao)

Fix $A \subseteq G$ and suppose $H \leq G$ is contained in $\mathrm{Stab}_\epsilon(A)$. Then there is $D \subseteq G$, which is a union of right cosets of H, such that $|A \triangle D| \leq \epsilon|G|$.

Approximate subgroups

Fix a group G and a nonempty finite subset $S \subseteq G$.

Definition: S is a k-approximate subgroup if $1 \in S$, $S = S^{-1}$, and S^2 is covered by k left translates of S.

Approximate subgroups

Fix a group G and a nonempty finite subset $S \subseteq G$.

Definition: S is a k-approximate subgroup if $1 \in S$, $S = S^{-1}$, and S^2 is covered by k left translates of S.

Theorem (Breuillard, Green, Tao; Hrushovski)

Suppose G has exponent r and S is a k-approximate subgroup. Then S^4 contains a subgroup H of size $\Omega_{r,k}(|S|)$.

Approximate subgroups

Fix a group G and a nonempty finite subset $S \subseteq G$.

Definition: S is a k-approximate subgroup if $1 \in S$, $S = S^{-1}$, and S^2 is covered by k left translates of S.

Theorem (Breuillard, Green, Tao; Hrushovski)

Suppose G has exponent r and S is a k-approximate subgroup. Then S^4 contains a subgroup H of size $\Omega_{r,k}(|S|)$.

Theorem (Tao)

If $|S^3| \leq k|S|$ then $(S \cup S^{-1})^2$ is a $O(k^{O(1)})$-approximate subgroup.

Corollary (weak Bogolyubov-Ruzsa)

Suppose G has exponent r, $S = S^{-1}$, and $|S^3| \leq k|S|$. Then S^8 contains a subgroup of size $\Omega_{r,k}(|S|)$.

VC-sets in abelian groups of bounded exponent

Theorem (C. 2018)

Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a subgroup $H \leq G$, of index $O_{r,d}((1/\epsilon)^{d+1})$, and*
- *a set $D \subseteq G$, which is a union of right cosets of H,*

such that $|A \triangle D| \leq \epsilon|G|$.

VC-sets in abelian groups of bounded exponent

Theorem (C. 2018)

Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a subgroup $H \leq G$, of index $O_{r,d}((1/\epsilon)^{d+1})$, and*
- *a set $D \subseteq G$, which is a union of right cosets of H,*

such that $|A \triangle D| \leq \epsilon |G|$.

Recall: If $H \leq G$ and $K = \bigcap_{g \in G} gHg^{-1}$ then $[G : K] \leq [G : H]!$.

VC-sets in abelian groups of bounded exponent

Theorem (C. 2018)

Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

* *a normal subgroup $H \leq G$, of index $2^{O_{r,d}((1/\epsilon)^{d+1})}$, and*
* *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon |G|$.

VC-sets in abelian groups of bounded exponent

Theorem (C. 2018)

Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a normal subgroup $H \leq G$, of index $2^{O_{r,d}((1/\epsilon)^{d+1})}$, and*
- *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon|G|$.

Moreover, there is a set $Z \subseteq G$, with $|Z| \leq \epsilon^{1/2}|G|$, such that for any $g \notin Z$, either $|gH \cap A| \leq \epsilon^{1/4}|H|$ or $|gH \cap A| \geq (1 - \epsilon^{1/4})|H|$.

VC-sets in abelian groups of bounded exponent

Theorem (C. 2018)

Suppose G is a finite group of exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

- *a normal subgroup $H \leq G$, of index $2^{O_{r,d}((1/\epsilon)^{d+1})}$, and*
- *a set $D \subseteq G$, which is a union of cosets of H,*

such that $|A \triangle D| \leq \epsilon |G|$.

Moreover, there is a set $Z \subseteq G$, with $|Z| \leq \epsilon^{1/2}|G|$, such that for any $g \notin Z$, either $|gH \cap A| \leq \epsilon^{1/4}|H|$ or $|gH \cap A| \geq (1 - \epsilon^{1/4})|H|$.

This yields a regular partition for the bipartite graph on (G, G) induced by $xy \in A$, in which the pieces are the cosets of H and the regular pairs have density within ϵ of 0 or 1.

Removing the bound on the exponent

AFZ: If $G = (\mathbb{Z}/p\mathbb{Z}, +)$ and $A = \{0, 1, \ldots, \frac{p-1}{2}\}$ then $\mathrm{VC}(A) \leq 3$.

Removing the bound on the exponent

AFZ: If $G = (\mathbb{Z}/p\mathbb{Z}, +)$ and $A = \{0, 1, \ldots, \frac{p-1}{2}\}$ then $\mathrm{VC}(A) \leq 3$.

But we cannot have $|A \triangle D| < \frac{1}{2}|G|$, where D is a union of cosets of a subgroup of $\mathbb{Z}/p\mathbb{Z}$ whose index is independent of p.

Bohr sets

Definition

Given a group H, a homomorphism $\tau \colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

Bohr sets

Definition

Given a group H, a homomorphism $\tau\colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

$B \subseteq H$ is a (δ, n)-Bohr set in H if $B = B^n_{\tau,\delta}$ for some τ.

Bohr sets

Definition

Given a group H, a homomorphism $\tau \colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

$B \subseteq H$ is a (δ, n)-Bohr set in H if $B = B^n_{\tau,\delta}$ for some τ.

Remarks:

- $B^n_{\delta,\tau}$ is closed under inverses and contains the identity.

Bohr sets

Definition

Given a group H, a homomorphism $\tau\colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

$B \subseteq H$ is a (δ, n)-Bohr set in H if $B = B^n_{\tau,\delta}$ for some τ.

Remarks:

- $B^n_{\delta,\tau}$ is closed under inverses and contains the identity.
- $B^n_{\delta,\tau} \cdot B^n_{\delta,\tau} \subseteq B^n_{2\delta,\tau}$.

Bohr sets

Definition

Given a group H, a homomorphism $\tau \colon H \to \mathbb{T}^n$, and some $\delta > 0$, set

$$B^n_{\delta,\tau} := \{x \in H : d(\tau(x), 0) < \delta\},$$

where d denotes the usual metric on $\mathbb{T}^n$, and 0 is the identity in $\mathbb{T}^n$.

$B \subseteq H$ is a (δ, n)-Bohr set in H if $B = B^n_{\tau,\delta}$ for some τ.

Remarks:

- $B^n_{\delta,\tau}$ is closed under inverses and contains the identity.
- $B^n_{\delta,\tau} \cdot B^n_{\delta,\tau} \subseteq B^n_{2\delta,\tau}$.
- Bohr sets in abelian groups contain large “coset progressions”.

Dense subsets of finite groups

Bogolyubov's Lemma (1939; see Ruzsa 1994)

Suppose G is a finite abelian group and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $2S - 2S$ contains a $(1/4, n)$-Bohr set $B \subseteq G$, with $n < (1/\alpha)^2$.

Dense subsets of finite groups

Bogolyubov's Lemma (1939; see Ruzsa 1994)

Suppose G is a finite abelian group and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $2S - 2S$ contains a $(1/4, n)$-Bohr set $B \subseteq G$, with $n < (1/\alpha)^2$.

Theorem (C. 2018)

Suppose G is a finite group and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then there are:

- *a normal subgroup $H \leq G$ of index $O_\alpha(1)$, and*
- *a (δ, n)-Bohr set $B \subseteq H$, with $\delta^{-1}, n \leq O_\alpha(1)$,*

such that $B \subseteq S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$.

Dense subsets of finite groups

Bogolyubov's Lemma (1939; see Ruzsa 1994)

Suppose G is a finite abelian group and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $2S - 2S$ contains a $(1/4, n)$-Bohr set $B \subseteq G$, with $n < (1/\alpha)^2$.

Theorem (C. 2018)

Suppose G is a finite group and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then there are:

* *a normal subgroup $H \leq G$ of index $O_\alpha(1)$, and*
* *a (δ, n)-Bohr set $B \subseteq H$, with $\delta^{-1}, n \leq O_\alpha(1)$,*

such that $B \subseteq S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.

The proof uses pseudofinite groups, specifically the "ultraproduct of counterexamples" method.

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume Th(G) is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume $\mathrm{Th}(G)$ is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

- "Pseudofinite Sanders-Croot-Sisask analysis": There is a countably type-definable, bounded-index, normal subgroup $K \leq G$ such that $K \subseteq S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.
  (Similar to Massicot-Wagner and Krupiński-Pillay.)

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume $\mathrm{Th}(G)$ is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

- "Pseudofinite Sanders-Croot-Sisask analysis": There is a countably type-definable, bounded-index, normal subgroup $K \leq G$ such that $K \subseteq S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.
  (Similar to Massicot-Wagner and Krupiński-Pillay.)
- Pillay: The connected component of the identity in G/K is abelian.
  (Nikolov, Schneider, Thom: True for any compactification of G.)

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume Th(G) is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

- "Pseudofinite Sanders-Croot-Sisask analysis": There is a countably type-definable, bounded-index, normal subgroup $K \leq G$ such that $K \subseteq S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$.
(Similar to Massicot-Wagner and Krupiński-Pillay.)
- Pillay: The connected component of the identity in G/K is abelian. (Nikolov, Schneider, Thom: True for any compactification of G.)
- Structure of compact groups: $K = \bigcap_{i=0}^{\infty} B_i$, where B_i is a (δ_i, n_i)-Bohr set in a definable finite-index normal $H_i \leq G$.

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume Th(G) is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

- "Pseudofinite Sanders-Croot-Sisask analysis": There is a countably type-definable, bounded-index, normal subgroup $K \leq G$ such that $K \subseteq S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$.
  (Similar to Massicot-Wagner and Krupiński-Pillay.)
- Pillay: The connected component of the identity in G/K is abelian.
  (Nikolov, Schneider, Thom: True for any compactification of G.)
- Structure of compact groups: $K = \bigcap_{i=0}^{\infty} B_i$, where B_i is a (δ_i, n_i)-Bohr set in a definable finite-index normal $H_i \leq G$.
- C.-Pillay-Terry: Replace B_i by a definable "approximate Bohr set".

Pseudofinite ingredients

Let G be a saturated expansion of a group, and assume Th(G) is pseudofinite. Suppose $S \subseteq G$ is definable and $\mu(S) > 0$, where μ is the normalized pseudofinite counting measure.

- "Pseudofinite Sanders-Croot-Sisask analysis": There is a countably type-definable, bounded-index, normal subgroup $K \leq G$ such that $K \subseteq S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$.
(Similar to Massicot-Wagner and Krupiński-Pillay.)
- Pillay: The connected component of the identity in G/K is abelian.
(Nikolov, Schneider, Thom: True for any compactification of G.)
- Structure of compact groups: $K = \bigcap_{i=0}^{\infty} B_i$, where B_i is a (δ_i, n_i)-Bohr set in a definable finite-index normal $H_i \leq G$.
- C.-Pillay-Terry: Replace B_i by a definable "approximate Bohr set".
- Alekseev, Glebskiĭ, Gordon: Approximate Bohr sets in finite groups contain large Bohr sets.

The abelian exponent of a group

The abelian exponent of a group G is the smallest $r \geq 1$ (if it exists) such that $H/[H, H]$ has exponent at most r for any normal $H \leq G$.

The abelian exponent of a group

The abelian exponent of a group G is the smallest $r \geq 1$ (if it exists) such that $H/[H, H]$ has exponent at most r for any normal $H \leq G$.

Facts:

- The abelian exponent of G is at most the exponent of G.

The abelian exponent of a group

The abelian exponent of a group G is the smallest $r \geq 1$ (if it exists) such that $H/[H, H]$ has exponent at most r for any normal $H \leq G$.

Facts:

- The abelian exponent of G is at most the exponent of G.
- If G is nonabelian and simple then it has abelian exponent 1.

The abelian exponent of a group

The abelian exponent of a group G is the smallest $r \geq 1$ (if it exists) such that $H/[H, H]$ has exponent at most r for any normal $H \leq G$.

Facts:

- The abelian exponent of G is at most the exponent of G.
- If G is nonabelian and simple then it has abelian exponent 1.
- If G has abelian exponent r and H has abelian exponent 1 then $G \times H$ has abelian exponent r.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Proof: Let $S_* = S^2 S^{-2} \cap (SS^{-1})^2 \cap S^{-2} S^2 \cap (S^{-1} S)^2$.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Proof: Let $S_* = S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.

- We have $B \subseteq S_*$, where B is a (δ, n)-Bohr set in a normal subgroup $H \leq G$, with $\delta^{-1}, n, [G : H] \leq O_\alpha(1)$.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Proof: Let $S_* = S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.

- We have $B \subseteq S_*$, where B is a (δ, n)-Bohr set in a normal subgroup $H \leq G$, with $\delta^{-1}, n, [G : H] \leq O_\alpha(1)$.
- B contains the kernel K of some homomorphism $\tau \colon H \to \mathbb{T}^n$.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Proof: Let $S_* = S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.

- We have $B \subseteq S_*$, where B is a (δ, n)-Bohr set in a normal subgroup $H \leq G$, with $\delta^{-1}, n, [G : H] \leq O_\alpha(1)$.
- B contains the kernel K of some homomorphism $\tau \colon H \to \mathbb{T}^n$.
- $H/K \leq \mathbb{T}^n$ is a finite abelian group of exponent at most r, and is generated by at most n elements. So $|H/K| \leq r^n \leq O_{r,\alpha}(1)$.

Bogolyubov's Lemma for bounded abelian exponent

Corollary

Suppose G is a finite group of abelian exponent r, and $S \subseteq G$ is such that $|S| \geq \alpha|G|$. Then $S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$ contains a normal subgroup of G of index $O_{r,\alpha}(1)$.

Proof: Let $S_* = S^2S^{-2} \cap (SS^{-1})^2 \cap S^{-2}S^2 \cap (S^{-1}S)^2$.

- We have $B \subseteq S_*$, where B is a (δ, n)-Bohr set in a normal subgroup $H \leq G$, with $\delta^{-1}, n, [G : H] \leq O_\alpha(1)$.
- B contains the kernel K of some homomorphism $\tau \colon H \to \mathbb{T}^n$.
- $H/K \leq \mathbb{T}^n$ is a finite abelian group of exponent at most r, and is generated by at most n elements. So $|H/K| \leq r^n \leq O_{r,\alpha}(1)$.
- Now $\bigcap_{g \in G} gKg^{-1}$ is the desired normal subgroup of G.

VC-sets and abelian exponent

Theorem (C. 2018)

Suppose G is a finite group of abelian exponent at most r and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$. Then, for any $\epsilon > 0$, there are:

* *a normal subgroup $H \leq G$, of index $O_{r,d,\epsilon}(1)$,*
* *a set $D \subseteq G$, which is a union of cosets of H, and*
* *a set $Z \subseteq G$, with $|Z| \leq \epsilon^{1/2}|G|$,*

such that

(i) $|A \triangle D| \leq \epsilon |G|$, *and*

(ii) *for any $g \notin Z$, either $|gH \cap A| \leq \epsilon^{1/4}|G|$ or $|gH \backslash A| \leq \epsilon^{1/4}|G|$.*

VC sets in nonabelian finite simple groups

Corollary

For any $d \geq 1$ and $\epsilon > 0$, there is $n = n(d, \epsilon)$ such that if G is a nonabelian finite simple group of size at least n, and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$, then $|A| < \epsilon|G|$ or $|A| > (1 - \epsilon)|G|$.

VC sets in nonabelian finite simple groups

Corollary

For any $d \geq 1$ and $\epsilon > 0$, there is $n = n(d, \epsilon)$ such that if G is a nonabelian finite simple group of size at least n, and $A \subseteq G$ is such that $\mathrm{VC}(A) \leq d$, then $|A| < \epsilon|G|$ or $|A| > (1 - \epsilon)|G|$.

Using work of Gowers on "quasirandom" groups, one can show

$$n(d, \epsilon) = 2^{O((90/\epsilon)^{6d})}.$$

thank you