Division rings with ranks

joint work with Daniel Palacin

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   - finite burden

Overview

Division rings

Division rings

ranked/superrosy

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Let D be a division ring carrying an ordinal-valued rank function among the definable sets in the imaginary expansion, i.e.

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2. The rank is preserved under definable bijections.
3. The Lascar inequalities: For a definable subgroup H of a definable group G we have that

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Example: See blackboard

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Remark

The U^b rank satisfies the above properties.

Easy consequences I

Let G and H be two definable groups and let $f : H \to G$ be a definable group morphism. Then

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- if $H < G$, then $\mathrm{rk}(H) = \mathrm{rk}(G)$ iff $[G : H] < \infty$.

Easy consequences II (DCC)

There is no infinite descending of definable groups

$$H_0 > H_1 > \cdots > H_n > \dots$$

each of them having infinite index in its predecessor.
(no infinite strictly descending chain of ordinals)

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In particular, for infinite subdivision rings, we obtain that every descending chain stabilizes after finitely many steps.

wide/negligible

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If there is no confusion we simply say that Y is wide or respectively negligible.

On the way to the main result

Lemma

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Proof uses Schlichtings's theorem.

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Proof uses Schlichtings's theorem.

Corollary

Let D be a superrosy division ring. If a definable group morphism from D^+ or $D^\times$ to D^+ has a negligible kernel, its image has finite index in D^+.

Main result

Theorem (H., Palacin)

A division ring with a superrosy theory has finite dimension over its center.

Division rings

weight 1

Definition

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Let p be a type and λ be a cardinal. Then weight of p is at least λ if there is a non-forking extension $\mathrm{tp}(a/A)$ of p and a sequence $(a_i : i < \lambda)$ independent over A such that for all $i < \lambda$, we have that $a \not\perp_A a_i$.

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weight 1: Let b be generic over A of weight 1.

$$b \underset{A}{\not\downarrow} a_0 \text{ and } b \underset{A}{\not\downarrow} a_1 \quad \Rightarrow \quad a_0 \underset{A}{\not\downarrow} a_1$$

Fact

Fact (Krupinski, Pillay)

Let G be a group with a simple theory of weight 1 and A be a parameter set. Then the non-generic elements over A form a subgroup.

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Corollary

Let D be a division ring with a simple theory of weight 1 and A be a parameter set. Then the non-generic elements over A form a subdivision ring.

Main result

Theorem (H., Palacin)

A division ring with a simple theory and a generic of weight one is a field.

Division rings

finite burden

Definition

Given a division ring D and a natural number n, we introduce the following property:

$(\dagger)_n$ For any definable subgroups $H_0, \ldots, H_n$ of D^+, there exists some $j \leq n$ such that $[\bigcap_{i \neq j} H_i : \bigcap_{i \leq n} H_i] < \infty$.

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Remark

A definable division ring of burden n satisfies $(\dagger)_n$.

Results (H., Palacin)

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- D has dimension at most n over any infinite definable subfield, in particular over its center.
- D has the DCC on definable subfields.
- If $n = 1$ (e. g. D is an inp-minimal division ring) then D is commutative.

Definable group actions

Fact (Hrushovski)

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Proposition (H., Palacin)

If F is a field satisfying $(\dagger)_n$, then any definable group of automorphisms acting definably on F has size at most n.

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Fact (Hrushovski)

Any definable group of automorphisms acting definably on a definable superstable field is trivial.

Proposition (H., Palacin)

If F is a field satisfying $(\dagger)_n$ and the algebraic closure of the prime field of F in F is infinite, then any definable group of automorphisms acting definably on F has size at most n.

Thank you