Does nonclassical truth impair mathematical reasoning?

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The proof-theoretic analysis of these two systems shows that while transfinite induction up to ε_0, a valid form of reasoning in PA, is valid in KF, it is not valid in PKF.

Axiomatising Kripke's theory of truth 1: KF

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¬1 $$\mathrm{Sent}_{\mathcal{L}_\mathsf{T}}(x), \neg \mathsf{T}x \Rightarrow \mathsf{T} \underset{\cdot}{\neg}\, x$$

¬2 $$\mathrm{Sent}_{\mathcal{L}_\mathsf{T}}(x), \mathsf{T} \underset{\cdot}{\neg}\, x \Rightarrow \neg \mathsf{T}x$$

Axiomatising Kripke's theory of truth 3: induction

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Induction must be given as a rule since the corresponding induction scheme, formulated with the truth predicate, is not sound in Kripke fixed point models.

Kripkean theories of truth

Adequacy criterion

Let $\mathcal{L}$ be a logic and S an $\mathcal{L}$-theory in the language L_T. S is adequate to the Kripkean conception of truth if for all $X \subseteq \omega$,

$$(\omega, X) \models_{\mathcal{L}} S \Leftrightarrow X \text{ is a Kripke fixed point.}$$

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Theorem (Feferman 1991, Halbach and Horsten 2006)

For all $X \subseteq \omega$,

$$\begin{aligned} &(\omega, X) \models \mathsf{KF} \\ \Leftrightarrow\ &(\omega, X) \models_{\mathrm{FDE}} \mathsf{PKF} \\ \Leftrightarrow\ &X \textit{ is a Kripke fixed point.} \end{aligned}$$

Transfinite induction

At the core of Peano Arithmetic, or PA, is the induction scheme.

$$((\forall m < n)\varphi(m) \to \varphi(n)) \to \forall n \varphi(n).$$

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As Gentzen showed, PA proves any instance of transfinite induction for $\varphi(x)$ in the language of arithmetic and $\alpha < \varepsilon$, i.e. $\textsc{ti}(<\varepsilon_0)$.

Induction and mathematical reasoning

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Similarly, when we reason using semantic predicates such as truth, we expand our language to include the truth predicate T, and at the same time we extend the induction scheme to include that predicate.

Transfinite induction, as a generalisation of induction, is much the same: it is a valid form of mathematical, non-semantic reasoning that remains valid when extended with additional theoretical vocabulary, including the truth predicate.

Transfinite induction in KF and PKF

KF is, in this sense, a mathematically faithful extension of PA. All of the theorems of PA are theorems of KF. But KF also proves all instances of the induction *axiom scheme* in the expanded language, and all instances of transfinite induction up to ε_0.

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This result shows that the classical recapture results available in nonclassical logic somehow fail to go far enough: they let us recapture the theorems of the original classical mathematical theory, but they do not recapture all the *patterns* of reasoning of classical mathematics, namely the open-ended ones.

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We will now close this gap by establishing that there are significant mathematical theorems in the gap, in particular a theorem concerning *indecomposable linear orderings*.

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3. The recursive (Δ^0_1) comprehension scheme.

Comparing KF and PKF

Let $\alpha < \Gamma_0$ be an ordinal. Then $\mathsf{ACA}^{<\alpha}$ is the system consisting of the basic axioms of PA^-, the full induction scheme, and for each $\beta < \alpha$, an axiom asserting that arithmetical comprehension can be iterated β-many times.

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Corollary

1. $\mathsf{RCA} + \text{INDEC}$ *is proof-theoretically reducible to* KF, *and* $\mathsf{RCA} + \text{INDEC} \equiv_{\mathcal{A}} \mathsf{KF}$.
2. $\mathsf{RCA} + \text{INDEC}$ *is not proof-theoretically reducible to* PKF.

INDEC is recoverable in KF, but not in PKF, so there really are significant mathematical theorems in the proof-theoretic gap between the classical and nonclassical theories of Kripkean truth.

Thank you!

Open questions

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The main barrier to further results is the lack of proof-theoretic analyses of nonclassical axiomatic theories of truth.

> *Only a proof-theoretic analysis of the nonclassical systems will provide a reliable assessment of the costs of adopting a nonclassical logic. (Halbach 2011, p. 244)*

Σ^1_1 choice and iterated predicative comprehension

The axiom of Σ^1_1 choice consists of the following scheme:

$$\forall n \exists Y \eta(n, Y) \rightarrow \exists Z \forall n \eta(n, (Z)_n)$$

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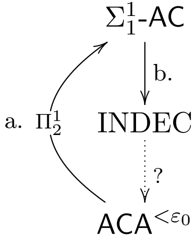
Theorem (Friedman 1967)

Σ^1_1-AC *is* Π^1_2 *conservative over* $\mathsf{ACA}^{<\varepsilon_0}$.

Theorem (Montalbán 2006)

Σ^1_1-AC *implies* INDEC.

Summary of implications



a. Friedman (1967): $\Sigma^1_1\text{-}\mathsf{AC} \equiv_{\Pi^1_2} \mathsf{ACA}^{<\varepsilon_0}$.

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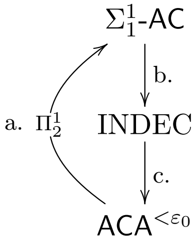

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c. $\mathsf{RCA} + \mathrm{INDEC} \vdash \mathsf{ACA}^{<\varepsilon_0}$.

Jump iteration

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Theorem (Neeman 2008)

INDEC *implies* JI.

We will show that JI implies the axioms of $\mathsf{ACA}^{<\varepsilon_0}$.

Jump iteration implies the existence of H-sets

Lemma

Fix an ordinal $\gamma < \varepsilon_0$. $\mathsf{RCA} + \mathrm{JI}$ *proves that for all* $X \subseteq \mathbb{N}$, H^X_γ *exists.*

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S. Feferman. Reflecting on incompleteness. *Journal of Symbolic Logic*, 56(1):1–49, March 1991. doi: 10.2307/2274902.

H. M. Friedman. *Subsystems of Set Theory and Analysis*. PhD thesis, Massachusetts Institute of Technology, 1967.

V. Halbach. *Axiomatic Theories of Truth*. Cambridge University Press, 2011. doi: 10.1017/CBO9780511921049.

V. Halbach and L. Horsten. Axiomatizing Kripke's theory of truth. *The Journal of Symbolic Logic*, 71(2):677–712, June 2006. doi: 10.2178/jsl/1146620166.

P. Jullien. *Contribution à l'étude des types d'ordre dispersés*. PhD thesis, Université d'Aix-Marseille, 1969.

A. Montalbán. Indecomposable linear orderings and hyperarithmetical analysis. *Journal of Mathematical Logic*, 6:89–120, 2006. doi: 10.1142/S0219061306000517.

References II

I. Neeman. The strength of Jullien's indecomposability theorem. *Journal of Mathematical Logic*, 8(1):93–119, 2008. doi: 10.1142/S0219061308000725.

I. Neeman. Necessary use of Σ^1_1 induction in a reversal. *The Journal of Symbolic Logic*, 76(2):561–574, 2011. doi: 10.2178/jsl/1305810764.