Kreisel's Problem

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Kreisel's problem (KP):

Is any logical consequence of ZFC ensured to be true?

The purpose of this talk is to provide an answer.

SUMMARY

1. The original problem (OP) raised by Georg Kreisel and George Boolos
2. Kreisel's and Boolos' solutions to (OP), transposed to (KP)
3. Another way of framing (KP)
4. The corresponding solution to (KP)
5. Extension: a new modal logic for models of ZFC.

The original problem (OP)

Originally, Georg Kreisel ("Informal rigour and completeness proofs") and George Boolos ("Nominalist Platonism") did not raise (KP), but both raised **a slightly different problem, (OP):**

Given the language L of ZFC, can we be sure that any logically valid L-sentence is true?

Kreisel's answer is positive and appeals to the completeness theorem for first-order logic.

Boolos provides two positive answers, which resort to the reflection principle and to the completeness theorem, respectively.

Two views about (OP)

Kreisel and Boolos both set up (OP) at the level of the background set-theoretic universe, namely:

Is any L-sentence that is logically valid (i.e., true in any structure contained in the universe), **true in the universe**?

This formulation lays itself open to the following attack: Logical validity w.r.t. the universe makes perfect sense, but truth in the universe cannot be defined explicity.

In contrast with Kreisel-Boolos view, the **model-scaled view** is the semantical view that considers only L-structures or models (as opposed to the background universe).

In this view, it makes sense to say that an L-sentence is true in some L-structure, but it seems to make no sense at all to say that that sentence is logically valid w.r.t. some L-structure.

Predicament

	Kreisel-Boolos View	Model-Scaled View
Logical Validity	OK	?
Truth	?	OK

Two ways of framing Kreisel's original problem (OP)

The same predicament strikes the treatment of (KP), **to which we will now turn.**

Kreisel's and Boolos' respective answers to (OP) can be transposed and completed so as to provide answers to (KP).

Kreisel's solution to (OP) transposed to (KP)

For any sentence ϕ of $L = L(\text{ZFC})$,

ZFC $\vDash^+ \phi :=$ "ϕ is true in any set or class structure that satisfies ZFC"

$\phi_\in :=$ "α is true when the quantifiers in ϕ range over all sets and $\in$ is taken to be the real membership relation"

Kreisel-style solution:

ZFC $\vDash \phi$ entails ZFC $\vdash \phi$, which entails ZFC $\vDash^+ \phi$, which in turn entails $\phi_\in$: problem solved.

Shortcoming: One is led to consider that a sentence is true in this a **little bit special** model that is the universe of all sets.

Shortcomings of Kreisel's solution

- The universe can be regarded only as a potential totality, and, as a consequence, truth in the universe should not be regarded as determined for every sentence.
- In any case, truth in the universe cannot be handled exactly in the same way as truth in a given model, since no formal semantics can underpin both kinds of truth. Unless the universe is plunged with all other models into some further background universe —but then, precisely, it would cease to be **the** universe.
- Kreisel's proof is laid out in ZFC. But, ensuing from Löb's theorem:

  $$\mathsf{ZFC} \vdash \ulcorner \mathsf{ZFC} \vDash \phi \urcorner \rightarrow \phi \ \text{ only if } \ \mathsf{ZFC} \vdash \phi \ .$$

  In other words, "If ϕ is true in every model of ZFC, then ϕ" can be derived only if ϕ is already a theorem of ZFC.

  Any solution à la Kreisel seems to be trivialized.

Another solution has to be found.

Boolos' solutions to (OP)

Boolos as well remarks that, oddly enough, logical validity does not guarantee truth. He suggests two ways out:

1. introducing the notion of "supervalidity" (as expressed by a monadic second-order sentence);
2. the reflection principle in ZFC.

The first solution cannot be transposed to (KP) because there is no clear way of defining the notion of being a "superconsequence of ZFC."

The second solution cannot as such be transposed to (KP), because the reflection principle deals with finite conjunctions only.

Transposing Boolos' second solution requires extending of the reflection principle through the addition of a satisfaction predicate $\mathrm{Sat}(u, v)$ and a truth predicate $\mathrm{Tr}(u)$ to the language L of ZFC.

Boolos' second solution to (OP) transposed to (KP)

Assume a usual set-theoretic coding of syntax.
For any formula of L, let $\ulcorner\phi\urcorner$ the set that codes ϕ.

Form(x):= "x is (the code of) a formula"
Sent(x):= "x is a sentence"
Ax(x):= "x is an axiom of ZFC"
Assign(y):= "y is a map with domain the set of (all codes for) the variable symbols"

Sat($\ulcorner\phi\urcorner, s$):="$s$ is an assignment for the variables of L which satisfies ϕ in V"

The formulas ϕ for which one has Sat($\ulcorner\phi\urcorner, s$) should be the original formulas of L not containing 'Sat', so that no paradox arises.

Boolos' solution transposed (cont'd)

Axioms for Sat and Tr:

1. $\forall x \forall y(\mathrm{Sat}(x, y) \to \mathrm{Form}(x) \wedge \mathrm{Assign}(y))$;
2. the usual inductive clauses for satisfaction:
   - $\mathrm{Sat}(v_1 \ulcorner\in\urcorner v_2, s) \leftrightarrow s(v_1) \in s(v_2)$
   - $\mathrm{Sat}(v_1 \ulcorner=\urcorner v_2, s) \leftrightarrow s(v_1) = s(v_2)$
   - $\mathrm{Sat}(\ulcorner\neg\urcorner u, s) \leftrightarrow \neg\mathrm{Sat}(u, s)$,
     $\mathrm{Sat}(u \ulcorner\vee\urcorner u', s) \leftrightarrow (\mathrm{Sat}(u, s) \vee \mathrm{Sat}(u', s))$
   - $\mathrm{Sat}((\ulcorner\exists\urcorner v)u, s) \leftrightarrow \exists x \mathrm{Sat}(u, s[x/s(v)])$
3. $\mathrm{Tr}(u) := (\mathrm{Sent}(u) \wedge \forall y(\mathrm{Assign}(y) \to \mathrm{Sat}(u, y)))$.

Let **ZFCS** be the resulting system in $L^+ = L \cup \{\mathrm{Sat}, \mathrm{Tr}\}$, where the replacement axiom and the separation axiom are extended to include formulas in which 'Sat' or 'Tr' occurs.

Boolos' solution transposed (cont'd)

It is well-known that semantic notions about L can be formalized within L. This formalization readily extends to L^+.

In particular, there is a formula

$$\Theta(A, u) := \ulcorner A \vDash \sigma \urcorner$$

in L^+ to the effect that A is a structure for L^+, u is $\ulcorner \sigma \urcorner$ for some sentence σ of L^+, and $A \vDash \sigma$.

Moreover, the proof of the reflection principle for ZFC readily extends to ZFCS.

Boolos-style solution: Let ϕ some true sentence in V. Applying the reflection principle to $(\phi \wedge \forall u(\mathrm{Ax}(u) \rightarrow \mathrm{Tr}(u)))$, one gets:

$$\mathsf{ZFCS} \vdash \exists \beta (\phi \wedge \forall u(\mathrm{Ax}(u) \rightarrow \mathrm{Tr}(u)))^{V_\beta} \ .$$

But $\mathsf{ZFCS} \vdash \forall A(\psi^A \leftrightarrow \ulcorner A \vDash \psi \urcorner)$. Hence:

$$\mathsf{ZFCS} \vdash \exists \beta \ulcorner V_\beta \vDash \phi \wedge \forall u(\mathrm{Ax}(u) \rightarrow \mathrm{Tr}(u)) \urcorner \ .$$

And since $\mathsf{ZFCS} \vdash \ulcorner A \vDash \mathrm{Tr}(\ulcorner \sigma \urcorner) \urcorner \rightarrow \ulcorner A \vDash \sigma \urcorner$, one has:

$$\mathsf{ZFCS} \vdash \exists \beta \ulcorner V_\beta \vDash \mathsf{ZFC} + \phi \urcorner \ .$$

Now, suppose that ϕ is not true. Then ZFCS proves that $V_\beta \vDash \mathsf{ZFC} + \neg\phi$ for some β, and so that ϕ is not a logical consequence of ZFC. By contraposition, ZFCS proves any logical consequence of ZFC to be true ("true" in the sense of 'Tr', which has been defined in L^+ but is not definable in L, owing to Tarski's theorem on the undefinability of truth).

Shortcoming of Boolos' solution

Boolos' solution basically lies in the fact that:

$$\mathsf{ZFCS} \vdash \ulcorner \mathsf{ZFC} \vDash \phi \urcorner \rightarrow \mathsf{Tr}(\ulcorner \phi \urcorner) \ .$$

However, ZFCS is significantly stronger than ZFC, since, as just shown, it proves Con(ZFC).

One should argue just from within ZFC. Indeed, the question naturally arises as to whether such a logical consequence of ZFCS as $(\ulcorner \mathsf{ZFC} \vDash \phi \urcorner \rightarrow \mathsf{Tr}(\ulcorner \phi \urcorner))$ is true itself.

The answer to (KP) has just been pushed back up a level.

Another solution has to be found.

Summary

	Kreisel's solution	Boolos' solution
Truth in the universe	Informal	Formalized through a satisfaction predicate added to the language of ZFC
Solution to (KP)	Trivialized by Löb's theorem	Requires shifting to ZFCS, a proper extension of ZFC

(KP): Is any logical consequence of ZFC true?

Toward another way of framing (KP)

Boolos and Kreisel considered two kinds of truth:

- truth in a set structure;
- and truth in the background universe.

It is clearer to deal with only a single kind of truth: The notion of truth that occurs in the definition of being a logical consequence of ZFC, as truth in any structure for the language, should be the same as that about which it is asked whether or not it is ensured by being a logical consequence of ZFC.

Justification of the new way

As opposed to the difficulties that affect the Kreisel's as well as Boolos' solutions, there is in fact a structure in which all the sentences of the language of ZFC are ensured to have formalized truth conditions and in which all the sentences derivable in ZFC are ensured to be true: **namely, a model of ZFC!**

So the natural way to go is to frame (KP) **at the level of models of ZFC,** so that any definition of truth in the universe becomes unnecessary. **This will be the principle of the solution proposed in this talk.**

Obviously, the counterpart of that new option is the need to define what it means, for a sentence of L to be, **relatively to some model of ZFC**, a logical consequence of ZFC.

Justification of the new way (cont'd)

Both Kreisel and Boolos tended to consider the background set-theoretic universe as a kind of monster model (the intended model of the metatheory).

Let's turn things around, by turning each model of ZFC into a universe in its own right.

Such a **model-scaled construal of (KP)** is actually compatible with the "Multiverse View," yet does not force its endorsement.

The Multiverse View amounts to holding that there are as many universes as there are models of ZFC.

The Model Scaled View which I advocate consists in identifying all models of ZFC with as many universes.

Models as universes

Standard coding. The code of any formula ϕ of L consists in a sequence $\ulcorner\phi\urcorner$ of numerals, and gives rise in any model M of ZFC to an interpretation $\ulcorner\phi\urcorner^M$, where each numeral of the sequence is interpreted by the corresponding integer of M.

The main notions in the metatheory of ZFC ("being a formula," "being a proof in ZFC," "being a model of a sentence") **can be formalized within the first-order language L of ZFC.**

For instance, it is possible to define in L the predicate 'For(x)' to the effect that x encodes the construction of a formula of L.

An M-**formula** is an object a in $|M|$ such that $M \vDash$ For$(x)[a]$. Any $\ulcorner\phi\urcorner^M$ is an M-formula, but the converse is not true.

Models as universes (cont'd)

A model $M = \langle M, \in_M \rangle$ of ZFC is ω-**standard** if $\in_M$ is transitive and well-orders all the finite ordinals of M.

If M is ω-standard, the M-formulas (resp. the M-proofs) are in a 1-1 correspondence with the genuine formulas (resp. the proofs) of ZFC.

If not, some M-formulas and M-proofs fail to correspond to any formula or proof of ZFC.

Models as universes (cont'd)

Let M be a model of ZFC, and N an element of $|M|$ such that $M \vDash \ulcorner$"N is a structure for L"$\urcorner$.

This implies that there exists $|N|, E^N \in |M|$ such that $M \vDash (N = \langle |N|, E^N \rangle \wedge E^N \subseteq |N| \times |N|)$.

One then defines:

$$|N_M| := \{x \in |M| : M \vDash x \in |N|\}$$

$$E^N_M := \{(x, y) \in |N_M| \times |N_M| : M \vDash (x, y) \in E^N\}$$

The structure $N_M := \langle |N_M|, E^N_M \rangle$ is the "replica" of N in M.

In case M is a transitive $\in$-model ($\in_M = \in \restriction |M|$ and $x \in |M| \to x \subseteq |M|$), $N_M \simeq N$.

Lemma

For any sentence ϕ of L and any model M of ZFC, one has:
$M \models \ulcorner N \models \phi \urcorner$ *iff* $N_M \models \phi$.

Theorem (Suzuki-Wilmers (1971), Schlipf (1978))

Let M be a model of ZFC. Then there exists $N \in |M|$ such that $N_M \models ZFC$ *(but not necessarily:* $M \models \ulcorner N \models ZFC \urcorner$*).*

Proof. Two cases:

- M is ω-standard.
  The very existence of M implies that ZFC is consistent (so, no proof of '$0 = 1$'). By hypothesis, there are no more proofs according to M than there are in reality, so $M \vDash \text{Con(ZFC)}$. And since the completeness theorem is true in M, the conclusion follows.

- M is not ω-standard. The idea is to index all the axioms of ZFC by some nonstandard ordinal of M, so that the reflection principle can be applied to what M thinks to be a finite conjunction of axioms.
  Suppose $M \vDash \neg\exists\alpha \ulcorner V_\alpha \vDash \text{ZFC}\urcorner$. Given $(A_i)_{i\in\mathbb{N}}$ a recursive enumeration of the axioms of ZFC, one gets, by compactness: $M \nvDash \forall n \exists\alpha \ulcorner V_\alpha \vDash A_0 \wedge A_1 \wedge \ldots \wedge A_n\urcorner$. Given the L-formula $\chi(n) := \exists\alpha \ulcorner V_\alpha \vDash A_0 \wedge A_1 \wedge \ldots \wedge A_n\urcorner$, there exists $n_0 \in \omega^M$ such that $M \vDash \neg\chi(n_0) \wedge \forall n < n_0\ \chi(n)$. Owing to the reflection principle, n_0 has to be a nonstandard integer of M. But $M \vDash \chi(n_0 - 1)$, and $n_0 - 1$ is also nonstandard.

Definition

An **internal model** of ZFC is any model of ZFC of the form N_M, for some model M of ZFC .

The previous result ensures that any model M of ZFC has internal models. Hence it becomes possible to define **logical consequence (from ZFC) w.r.t. some model** M.

Two special models:

- Shepherdson's minimal model M_0 of ZFC.
  All internal models of M_0 are nonstandard, and M_0 faithfully recognizes them to be so.
- Any model M^* of ZFC + ¬Con(ZFC).
  M^* does have internal models, but from the point of view of M^* they satisfy at most a finite number of the axioms of ZFC (only, this number is nonstandard).

From a set-theoretical point of view

N_M can be described as N as seen from M's point of view.

More generally, one is justified in considering any model of set theory, not only as a **domain**, that is as a place of evaluation of formal sentences, but also as a **point of view**, that is as a background universe in its own right.

This does not detract from the absolute point of view of the real universe, which is but the semantic counterpart of the fact that the analysis is kept within the limits of ZFC.

From a set-theoretical point of view (cont'd)

Viewing models as "points of view" only catches up with a well-established tradition dating back to Skolem's paradox.

Any member a of a model M of ZFC gives rise to the set

$$a^* = \{x \in |M| : x \in_M a\} \ .$$

The set a^* is nothing but a **as seen from** M**'s point of view**, even though a^* does not necessarily belong to M.
The relativity phenomenon in which Skolem's paradox is grounded is "the discrepancy between M's assessment of a and a's (or rather, a^*'s) true status." (I. Jané, "Reflections on Skolem's relativity of set-theoretical concepts").

The notion of point of view corresponds to the set-theoretic operation $(M, N \in |M|) \mapsto N_M$.

To sum up. While Kreisel and Boolos referred to the universe as being a kind of model, any model of ZFC can be looked at as being a surrogate universe from the point of view of which other models of ZFC can be considered.

Remark

The axiom of foundation is not violated, because an internal model N_M does not necessarily coincide with the element N of $|M|$.

Starting with a model M_0, there is $M_1 \in |M|$ such that $\mathbf{M_1'} := (\mathbf{M_1})_\mathbf{M} \models \mathrm{ZFC}$.

Then there is $(M_2)_{M_1'}$ **with** M_2 **belonging to** M_1'**, but not necessarily to** M_1 —so that any infinite descending $\in$-chain

$$\ldots |M_2| \in |M_1| \in |M|$$

is avoided in the end.

Depth of logical consequence?

Definition

An L-sentence is a **2-logical consequence** of ZFC if it is true in any internal model of ZFC.

In fact, 2-logical consequences and logical consequences of ZFC turn out to collapse:

Proposition

Let ϕ be a sentence of L. Then ϕ is a 2-logical consequence of ZFC iff it is a logical consequence of ZFC.

Definition

Let ϕ be an L-sentence and M be a model of ZFC.
ϕ is an M-**logical consequence of ZFC**, written $\mathbf{ZFC} \vDash_M \phi$, iff, for every $N \in |M|$, $N_M \vDash \mathrm{ZFC}$ implies $N_M \vDash \phi$.

The intuitive meaning of $\mathrm{ZFC} \vDash_M \phi$ is that ϕ would be a logical consequence of ZFC, **were** M **the background universe**.

Definition

ϕ is called an **internal logical consequence of ZFC**, written $\mathbf{ZFC} \vDash^{\mathbf{i}} \phi$, iff $\mathrm{ZFC} \vDash_M \phi$ for any model M of ZFC.

The intuitive meaning of $\mathrm{ZFC} \vDash^{\mathrm{i}} \phi$, then, is that ϕ is a logical consequence of ZFC from the points of view of all models.

Relationship between ZFC $\vDash_M \phi$ and $M \vDash \phi$

Let θ be the first strongly inaccessible ordinal. By a result of Montague and Vaught, there exists $\theta^* < \theta$ such that $\langle V_{\theta^*}, \in\rangle \equiv \langle V_\theta, \in\rangle$, and $(V_{\theta^*})_{V_\theta} = V_{\theta^*}$.

Consequently, ZFC $\vDash_{V_\theta} \phi$ implies $V_\theta \vDash \phi$.

Let's call a cardinal γ a **universe cardinal** iff $V_\gamma \vDash$ ZFC, and let γ_0 be the least universe cardinal.
The **weak axiom of universes** is the sentence WAU saying that "there are unboundedly many universe cardinals."
For κ inaccessible, $V_\kappa \vDash$ ZFC + WAU.
But, by minimality, $V_{\gamma_0} \nvDash$ WAU, and in fact $V_\kappa \vDash \ulcorner V_{\gamma_0} \nvDash \text{WAU}\urcorner$.

Consequently, $M \vDash \phi$ does not generally entail ZFC $\vDash_M \phi$.

Theorem

Let ϕ be an L-sentence and M be a model of ZFC such that $ZFC \vDash_M \phi$. Then $M \vDash \phi$.

Proof.

Let's suppose that $M \nvDash \phi$. This proves that ZFC $+ \neg\phi$ is consistent. The proof of the previous theorem can then be rewritten, with ZFC $+ \neg\phi$ replacing ZFC: there exists $N \in |M|$ such that $N_M \vDash$ ZFC $+ \neg\phi$, hence ZFC $\nvDash_M \phi$. □

Corollary

Let ϕ be an L-sentence. Then:

$$ZFC \vDash \phi \text{ iff } ZFC \vDash^i \phi$$

Proof.

The Theorem guarantees that ZFC $\vDash^{\mathrm{i}} \phi$ implies ZFC $\vDash \phi$. Conversely, suppose that ϕ is a logical consequence of ZFC. Then, in particular, $N_M \vDash \phi$ for any internal model N_M of ZFC, for any model M of ZFC. □

Back to (KP)

	Kreisel's Solution	Boolos' Solution	Model-Scaled Solution	Generalization to every M
ϕ is a logical consequence of ZFC	ZFC $\models^{+} \phi$	ZFC $\models \phi$	ZFC $\models_M \phi$	ZFC $\models^{\text{i}} \phi$
ϕ is true	ϕ is informally true	$\ulcorner\phi\urcorner$ is in the extension of the truth predicate added to L	ϕ is true in M	ϕ is true in every M
Answer to (KP)	Yes	Yes	No in general	Yes

(The last column of the table above is but the generalization to every M of the model-scaled view relativized to some model M of ZFC, as expressed by the previous column.)

Remark

The class of all models internal to M is not definable over M.

(This is because n is a nonstandard integer of M if and only if whenever $M \vDash \ulcorner N \vDash$ the first n axioms of ZFC$\urcorner$, N_M is a model of ZFC. But M cannot define its nonstandard integers.)

Proposition

Let M be a model of ZFC. We define the **standard system of** M *as being the set of the standard truncatures of all M's real numbers:*
$St(M) = \{st(A) : A \in |M|, M \vDash A \subseteq \omega\}$, *where*
$st(A) = \{n \in \mathbb{N} : M \vDash \overline{n} \in A\}$.
There is $N \in |M|$ such that $N_M \equiv M$ iff $\mathbf{Th}(M) \in \mathbf{St}(M)$.
In that case, being an M-logical consequence of ZFC ensures truth in M.

The criterion given by the previous proposition really divides the spectrum of all models into two camps:

- Any full standard model of second-order set theory contains every real, and in particular its own theory.
- On the other hand, the theory of any pointwise definable model M of ZFC cannot be in M's standard system. (Otherwise, you can mimick the Liar paradox in M.)

The natural step to take to strengthen the previous proposition is to require that some internal model is an **elementary substructure** of the original one.
Actually, the set of sentences true in (M, V_α^M) is too big to be a set in M. The best approximation of the existence of an internal elementary substructure is:

Proposition

Let M be a model of ZFC and α an ordinal of M.
Then there exists $N \in |M|$ such that

- $V_\alpha^M \subseteq |N|$ *and*
- $(N_M, V_\alpha^M) \equiv (M, V_\alpha^M)$

iff there exists $s \in |M|, s : (V_\alpha^M)^{<\omega^M} \to \wp(\omega^M)$ such that

$$\forall \vec{a} \in V_\alpha^M \quad st(s(\vec{a})) = Th(M, \vec{a}) \ .$$

Again, the criterion given by the proposition above really divides the spectrum of all models into two camps:

- Obviously, the minimal model M_0 does not have any internal model N_M as an elementary substructure.
- On the contrary, any recursively saturated model M of ZFC has one.
  (Consider the type composed of all formulas $\phi_n(x) =$ " any tuple of V_x satisfies any of the first n formulas of L in V_x exactly when it satisfies it in M ".)

Remark

The previous results can be extended to set theories stronger than ZFC, in particular to Morse-Kelley set theory.
Some results can also be found about set theories weaker than ZFC, in particular Kripke-Platek set theory with urelemente.

Going modal

The idea is to think of any **internal model N_M of M** as being a **possible world accessible from M**.

Definition

M' is **accessible from M** iff M' is (isomorphic to) some model of ZFC internal to M.

The difference with Hamkins-Löwe's "modal logic of forcing" is that the accessibility relation works downward instead of going upward.

Possibility being truth in some accessible possible world, a natural additional semantical clause is:

Definition

Let M be a model of ZFC and ϕ a sentence of the language L of ZFC.
$M \models \Diamond\phi$ iff there is $N \in |M|$ such that $N_M \models \mathrm{ZFC} + \phi$.

Proposition

There is no formula $P(x)$ of L such that, for any sentence ϕ of L and any model M of ZFC, $M \vDash \Diamond\phi$ iff $M \vDash P(\overline{n_\phi})$.

Proof.

By virtue of the fixed point theorem, there is a sentence ϕ such that $\phi = \neg P(\overline{n(\phi)})$. $M \vDash \phi$ implies $M \vDash \Diamond\phi$ which implies $M \vDash \neg\phi$, so ZFC $=$ ZFC $+ \neg\phi$. Now, suppose there is a model M of ZFC. One has $M \vDash P(\overline{n_\phi})$, hence the existence of some internal model N_M of ZFC $+ \neg\phi + \phi$: contradiction. □

However, adding directly a modal operator to the language L of ZFC is not an option.

Propositional modal logic

The language of propositional modal logic is generated from the language of propositional logic by adding a "necessity operator" $\Box$ and a "possibility operator" $\Diamond$.

The two modal operators are interdefinable: $\Box p := \neg\Diamond\neg p$, $\Diamond p := \neg\Box\neg p$.

A formal system of modal logic is said to be **normal** if it includes:

- all tautologies of propositional calculus;
- the axiom **K**, namely $\Box(p \to q) \to (\Box p \to \Box q)$;
- the rule of uniform substitution (if $A(p_1, \ldots, p_n)$ is a theorem, so is $A[\beta_k/p_k]$) $(k = 1, \ldots, n)$;
- the rule of *modus ponens*;
- the rule of necessitation (if A is a theorem, so is $\Box A$).

Propositional modal logic (cont'd)

Important modal axioms are:

- $\Box p \rightarrow p$, or equivalently $p \rightarrow \Diamond p$ (**T**)
- $\Box\Box p \rightarrow \Box p$, or equivalently $\Diamond\Diamond p \rightarrow \Diamond p$ (**4**)
- $\Diamond p \rightarrow \Box\Diamond p$ (**5**).

S4 is the least normal modal system that contains the axioms **T** and **4**. The system S5 is S4 + **5**.

Semantically, possibility intuitively means truth in at least one accessible possible world, and necessity intuitively means truth in all accessible possible worlds.

Interpretations of propositional modal logic into ZFC

L' = language of propositional modal logic ($\Box$ = necessity operator).
L = language of ZFC.

An **interpretation** i **of** L' **into** L is a map that assigns to each propositional letter p an arbitrary sentence of L.

For any such interpretation i, any structure M for L and any modal formula A, "$\mathbf{M} \vDash \mathbf{i}(\mathbf{A})$" is defined inductively as follows:

- $M \vDash i(\neg A)$ iff $M \nvDash i(A)$
- $M \vDash i((A \wedge B))$ iff $M \vDash i(A)$ and $M \vDash i(B)$
- $M \vDash i(\Diamond A)$ iff there is an internal model N_M of M such that $N_M \vDash i(A)$
- $M \vDash i(\Box A)$ iff ZFC $\vDash_M i(A)$.

Definition

Given a modal proposition A and a model M of ZFC, A is **modal-internally valid in** M iff, for any interpretation i of L' into L, $M \models i(A)$.

Definition

A modal proposition A is a **valid principle of internal modal logic** if it is modal-internally valid in any model of ZFC.

The set of all valid principles of internal modal logic is denoted by **IML**.

Proposition

IML is a normal modal logic.

Proposition

The modal reflexivity axiom **T** *,* $\Box p \to p$*, belongs to IML.*

Proof.

This is a direct consequence of our first theorem, to the effect that, for any model M of ZFC, there exists $N \in |M|$ such that $N_M \models$ ZFC. Just replace ZFC with ZFC $+ \neg i(p)$. So $M \not\models i(p)$ implies ZFC $\not\models_M i(p)$. □

Thus, IML encodes the existence of internal models (set-theoretic reflection) in the guise of the axiom **T** (modal reflexivity).

Remark

The axiom **T** *is IML-valid even though the accessibility relation in play is not reflexive.*

Definition

Given a certain class $\mathcal{K}$ of models of ZFC, a $\mathcal{K}$-**valid principle of internal modal logic** is a modal proposition A which is modal-internally valid in any member of $\mathcal{K}$.

This is written: $\mathcal{K} \vDash_{\mathbf{IML}} A$.

Theorem

Let $\mathcal{S}$ be the class of all standard models of ZFC. $\mathcal{S} \nvDash_{IML} \mathbf{5}$.

Theorem

Let $\mathcal{T}$ be the class of all transitive models of ZFC. $\mathcal{T} \vDash_{IML} S4$.

Downward stability

Since the interpretation of the necessity operator relies on the consideration of internal models, admissible classes of models of ZFC are those stable under internal models.

Definition

So let's say a class $\mathcal{K}$ of models of ZFC is **weakly downward stable** if, for every $M \in \mathcal{K}$, there exists $N \in |M|$ such that $N_M \in \mathcal{K}$.

It is **strongly downward stable** if, for every $M \in \mathcal{K}$ and every $N \in |M|$, $N_M \vDash \text{ZFC}$ implies $N_M \in \mathcal{K}$.

Lemma

The class $\mathcal{S}$ of all standard models of ZFC and the class $\mathcal{T}$ of all transitive models of ZFC are not weakly downward stable.

Lemma

The class $\mathcal{R}$ of all countable recursively saturated models of ZFC and the class $\mathcal{N}$ of all non-ω-standard models of ZFC are both strongly downward stable.

Remark: The class $\mathcal{R}$ has been studied by Victoria Gitman and Joel Hamkins, and proved to be a model of "the multiverse axioms."

Theorem

$\mathcal{R} \models_{IML} S4$.

Remark: This result uses the equivalence between "countable recursively saturated" and "countable resplendent."

Proof.

If ϕ is true in some internal model α_{N_M} of some internal model N_M of some model M, then ϕ is true in some internal model of M.

Let σ be a finite fragment of ZFC + ϕ. Since $N_M \models \ulcorner \alpha \models \sigma \urcorner$,

$$M \models \ulcorner\ N \models \ulcorner \langle |\alpha|, \in^{\alpha} \rangle \models \sigma \urcorner\ \urcorner$$

$$M \models \langle |\alpha|_N, (\in^{\alpha})_N \rangle \models \sigma \urcorner$$

which translates into

$$\langle M, |\alpha|_{N_M}, (\in_{\alpha})_{N_M} \rangle \models \sigma^*(P, R)\ .$$

So M can be expanded to a model of any finite fragment of (ZFC + ϕ)*.
Now, any resplendent L-structure M, some elementary extension N of which can be expanded to a model of a recursive theory T in $L(P, R)$, can itself be expanded to a model of T.
Owing to M's resplendency, M can be expanded to a model of (ZFC + ϕ)* in $L(P, R)$. The conclusion follows. □

Definition

A modal theory Λ is **IML-complete w.r.t. a class $\mathcal{K}$ of models of ZFC** if, for any modal proposition A,

$$\Lambda \vdash A \text{ iff } M \vDash_{\mathsf{IML}} A \quad \text{for every } M \in \mathcal{K}.$$

Theorem

S4 is IML-complete w.r.t. $\mathcal{R}$.

Thus, the modal axiom **4** encodes resplendency, as we saw that the axiom **T** encodes internal reflection.

Conclusion

1. Any model of set theory can be seen as a local universe, because it can be shown to embrace internal models. Not only truth in any given model of ZFC, **but also logical consequence of ZFC w.r.t. any such model** make sense after all.
2. A model-scaled treatment of **Kreisel's Problem** ("Are logical consequences of ZFC, true?") has to be favored, because it does not resort to any informal notion of truth in the background universe, and does not exceed the limits of ZFC either.
3. Moreover, the study of internal models of ZFC, or "set-theoretic prospecting," provides fine-grained results —whether in purely set-theoretic ones or in modal ones.

THANK YOU !