“Proofs as Programs” Revisited

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Aim of This Talk

- The aim is to revisit Schwichtenberg's works by focusing on parameter subsystems of Girard's F.

Proofs as Programs by Schwichtenberg

- Proofs are regarded as programs (Curry=Howard isomorphism)
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Theorem (Schwichtenberg90)

Let r be a closed term of type $\mathbb{N} \to \mathbb{N}$ in arithmetic. Then, there is m such that all $n \geq m$

$$|rn| \leq G_{D_0 D_1^{m+2} 0}(n).$$

(D_0, D_1 are the collapsing functions, and G is a slow growing hierarchy.)

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$$|\boldsymbol{rn}| \leq \boldsymbol{G}_{\boldsymbol{D_0 D_1^{m+2} 0(n)}}.$$

- Strategy for getting this result:

  1. Normalize a given term $\boldsymbol{rn}$ and measure the size of it. (We need $\boldsymbol{D_0 D_1^{m+2} 0(n)}$ here.)

  2. To climb down the "big" tree ordinal by the slow growing hierarchy using ideas by Wainer-Girard and Arai.

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  2. To climb down the "big" tree ordinal by the slow growing hierarchy using ideas by Wainer-Girard and Arai.

- The bound is sharp. A specific program of $\forall x \exists y A(x, y)$ has such a complexity.

- These arguments are implemented in Scheme.

Some Literatures

- Arai, A slow growing analogue to Buchholz' proof, 1991.
- Buchholz, An independence result for Π^1_1-$\mathbf{CA}$+$\mathbf{BI}$, 1987.
- Girard, Proof Theory and Logical Complexity, Vol 1, 1987.
  (Volume 2 is available: http://girard.perso.math.cnrs.fr/Archives4.html)
- Schwichtenberg, Proofs as Programs, 1990.
- Schwichtenberg and Wainer, Ordinal Bounds for Programs, 1994.

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- Two advantages of our approach:
  1. Our approach is simpler, smoother.
     - The syntax of F is very simple.
  2. This talk is about the weakest theory dealing with the type $\boldsymbol{N}$:

$$\boldsymbol{N} : \forall \alpha.((\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha)$$

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  $$\boldsymbol{N} : \forall\alpha.((\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha)$$

  3. It is possible to extend our result into stronger theories of inductive definitions, uniformly.
     - Typical example of the next level is Brouwer's ordinals:

  $$\boldsymbol{O} : \forall\alpha.((\boldsymbol{N} \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha)$$

- This is more direct, too.

  - Terms in $\boldsymbol{F}$ can be regarded as programs.

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- Examples of this direction:
  - Terui, "MacNeille completion and Buchholz' Omega rule for parameter-free second order logics", *CSL*, 2018.
  - Akiyoshi and Terui, "Strong normalization for the parameter-free polymorphic lambda calculus based on the Omega-rule", *FSCD*, 2016.
- Maybe, we could apply this method to another type theories, but I don't know...

Some Literatures

- Akiyoshi, “The Upperbound of the Length of the Reductions in a Subsystem of Girard’s F”, preprint, 2018.
- Akiyoshi, ““Proofs as Programs” in Parameter-Free Fragments of System F”, submitted, 2018.
- Akiyoshi, “A Formalization of Brouwer’s Argument for Bar Induction”, *WoLLIC*, 2018.
- Terui, “MacNeille completion and Buchholz’ Omega rule for parameter-free second order logics”, *CSL*, 2018.
- Akiyoshi, “An Ordinal-Free Proof of the Complete Cut-Elimination Theorem for Π^1_1-$\boldsymbol{CA} + \boldsymbol{BI}$ with the ω-rule”, *The Mints’ memorial issue of the IfCoLog Journal of Logics and their Applications*, 2017.
- Akiyoshi and Terui, “Strong Normalization for the Parameter-Free Polymorphic Lambda Calculus Based on the Ω-Rule”, *FSCD* 2016.
- Akiyoshi and Mints, “An Extension of the Omega-Rule”, *AML*, 2016.

Definition of Syntax

Definition

The *types* are defined by:

$$A, B ::= \alpha \mid A \Rightarrow B \mid \forall \alpha . A$$

where $\forall \alpha . A$ *is closed and* A *is* $\forall$*-free*.

Types in this set are "parameter-free".

Definition

Terms are defined as follows:

$$x^A \qquad (\lambda x^A . M^B)^{A \Rightarrow B} \qquad (M^{A \Rightarrow B} N^A)^B$$

$$(\Lambda \alpha . M^A)^{\forall \alpha . A} \qquad (M^{\forall \alpha . A} B)^{A[\alpha / B]}$$

with the standard proviso.

Examples

Examples of types in this language:

$$
\begin{array}{lcll}
\boldsymbol{N} & := & \forall\alpha.(\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \text{(natural numbers)} \\
\boldsymbol{T} & := & \forall\alpha.(\alpha \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \text{(binary trees)}
\end{array}
$$

Remark

Girard's maxim: Peano Arithmetic is (best viewed as) a theory of one inductive definition.

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But, we cannot express the following:

$$\begin{array}{ccll} \boldsymbol{L}(\boldsymbol{N}) & := & \forall\alpha.(\boldsymbol{N} \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \text{(lists over } \boldsymbol{N}\text{)} \\ \boldsymbol{O} & := & \forall\alpha.((\boldsymbol{N} \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) & \text{(Brouwer ordinals)} \end{array}$$

Remark

This kind of restriction originally goes back to Gaisi Takeuti's works in 1950's. Cf. his "On the fundamental conjecture of ***GLC*** *I-VI".*

Definition (Buchholz87)

The tree classes $\mathscr{T}_\sigma$ ($\sigma \leq \mathbf{2}$) are defined as follows:

- If $\alpha : \boldsymbol{I} \to \mathscr{T}_\sigma$ is a function with $\boldsymbol{I} : \emptyset, \{\mathbf{0}\}$, or $\mathscr{T}_\rho$ for some $\rho < \sigma$, then $\alpha \in \mathscr{T}_\sigma$.

Some notations.

1. $\mathbf{0}$ for $\alpha : \emptyset \to \mathscr{T}_\sigma$,
2. β^+ for $\alpha : \{\mathbf{0}\} \to \mathscr{T}_\sigma$ with $\alpha(\mathbf{0}) = \beta$.

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Remark

1. *$\mathscr{T}_\mathbf{0}$ is identified with $\mathbb{N}$ (the set of natural numbers) ,*
2. *$\mathscr{T}_\mathbf{1}$ is the set of countable trees.*

The operations of addition, multiplication, and exponentiation of trees are defined in the standard way. For example,

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma), (\alpha\times\beta)\times\gamma = \alpha\times(\beta\times\gamma), \boldsymbol{etc}\ldots$$

Collapsing Functions on Tree Ordinals

Let $\Omega_0 := \mathbb{N}$,
$\Omega_1 :=$ the set of countable tree ordinals.

Definition (Buchholz87, Arai91)

The collapsing functions $D_\sigma : \mathscr{T}_\nu \to \mathscr{T}_{\sigma+1}$ for $\sigma < \nu \le 2$ are defined as follows:

1. $\mathscr{D}_\sigma 0 := \Omega_\sigma$,
2. $\mathscr{D}_\sigma(\alpha+1) := (\mathscr{D}_\sigma(\alpha) \times (n+1))_{n\in\omega}$,
3. If $\rho \le \sigma$, then $\mathscr{D}_\sigma((\alpha_\xi)_{\xi\in\mathscr{T}_\rho}) := (\mathscr{D}_\sigma \alpha_\xi)_{\xi\in\mathscr{T}_\rho}$,
4. If $\sigma < \mu+1$, then $\mathscr{D}_\sigma((\alpha_\xi)_{\xi\in\mathscr{T}_{\mu+1}}) := (\mathscr{D}_\sigma \alpha_{\xi_n})_{n\in\omega}$ where $\xi_0 := \Omega_\mu, \xi_{n+1} := \mathscr{D}_\mu \alpha_{\xi_n}$.

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1. *In the last clause, the point is that the indexes ξ_n are tree ordinals.*
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3. *In this talk, we identify ordinals with its notations.*

- $\models^{a} M : A$ means "a term M has the size or complexity α".

Definition

The relation $\models^{a} M : A$ for $a \in \mathrm{T}$ is defined inductively as follows:

(*Var*) If $\models^{a} N_i : B_i$ for $i = 1, \ldots, m$ with $0 \leq m$, then $\models^{a+1} x\vec{N} : A$,

($<_1$) If $\models^{b} M : A$ and $b <_1 a$, then $\models^{a} M : A$,

(*abs*) If $\models^{a} N : C$, then $\models^{a+1} \lambda x.N : B \to C$,

(*Abs*) If $\models^{a} N : B$, then $\models^{a+1} \lambda\alpha.N : \forall\alpha.B$.

Remark

- *Idea is to define the "logical" domain over which we can quantify.*
- *In this definition, α could be just a natural number.*

The Relation $\vdash^{a}_{m} M : A$

Definition

The relation $\vdash^{a}_{m} M : A$ for $a \in \mathrm{T}$ and $m < \omega$ is defined by adding the following to $\models^{a}$:

(ω^{+}) If the following conditions are satisfied

i. $\mathrm{tp}(a) = \Omega_{1}, \vdash^{a^{-}}_{m} N : \forall\alpha.B$,

ii. $\forall z \in \mathrm{T}_{1} \forall K \in \Pi^{1}_{1}. \models^{z} K : B(\alpha)$ *implies* $\vdash^{a[z]}_{m} H_{z} : A$,

then $\vdash^{a}_{m} ND : A$,

(Cut) If $\vdash^{a}_{m} N$ with $lev(N) \leq m$ and and there is a sequence of terms $\vec{L}$ such that $\vdash^{a}_{m} L_{i}$ for $i = 1, \ldots, n$, then $\vdash^{a+1}_{m} N\vec{L}$.

Remark

- *H_{z} in the formulation of (ω^{+}) could depend on z.*
- *a is a limit of $a[0], a[1], \ldots$, hence could be infinite.*

Intuition of the ω^+-Rule

- Picture of the ω^+-rule:

$$\dfrac{\begin{array}{c}\vdots\\ \forall\alpha.\boldsymbol{B}\end{array} \qquad \dfrac{\dots \quad \dfrac{[\boldsymbol{K} : \boldsymbol{B}(\alpha)]}{\boldsymbol{B}[\alpha/\boldsymbol{D}]}\;\boldsymbol{S} \quad \dots}{\forall\alpha.\boldsymbol{B} \Rightarrow \boldsymbol{B}[\alpha/\boldsymbol{D}]}}{\boldsymbol{B}[\alpha/\boldsymbol{D}]}\;\omega^+$$

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- The right deduction is based on BHK-reading of $\Rightarrow$:

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- We can derive Comprehension using this:

$$\dfrac{\dfrac{[\forall\alpha.B]^1 \qquad \dfrac{\dots\ \dfrac{[K:B(\alpha)]}{B[\alpha/D]}\,S\ \dots}{\forall\alpha.B \Rightarrow B[\alpha/D]}}{B[\alpha/D]}\,\omega^+}{\forall\alpha.B \Rightarrow B[\alpha/D]}\,\rightarrow I, 1$$

Intuition of the ω^+-Rule

- We can compare the picture with the definition:

$$\dfrac{\begin{array}{c}\vdots\\ \forall\alpha.B\end{array} \quad \dfrac{\dots \dfrac{[K:B(\alpha)]}{B[\alpha/D]}\,S \dots}{\forall\alpha.B \Rightarrow B[\alpha/D]}}{B[\alpha/D]}\;\omega^+$$

(ω^+) If the following conditions are satisfied

i. $\mathrm{tp}(a) = \Omega_1, \vdash_m^{a^-} N : \forall\alpha.B,$

ii. $\forall z \in \mathrm{T}_1 \forall K \in \Pi_1^1.\ \models^z K : B(\alpha)$ *implies* $\vdash_m^{a[z]} H_z : B[\alpha/D],$

then $\vdash_m^a ND : B[\alpha/D]$.

Predicative Normalization

The following corresponds to predicative c.e. in infinitary proof-theory.

Lemma

There is an operation $\mathscr{D}_1$ on terms such that
If $\vdash^{\alpha}_{m+1} M : A$, then $\vdash^{\mathbf{D_1}a}_{m} \mathscr{D}_1(M) : A$.

Proof. The argument is more or less the same as the standard one. □

Idea of Impredicative Normalization

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{\forall\alpha.B} \qquad \dfrac{\ldots\ \dfrac{[K : B(\alpha)]}{B[\alpha/D]}\ S\ \ldots}{\forall\alpha.B \Rightarrow B[\alpha/D]}}{B[\alpha/D]}\ \omega^+$$

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  1. Normalize the proof of B. Let d be the result.

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- Strategy of normalization by induction on a given derivation:
  1. Normalize the proof of B. Let d be the result.
  2. Take the subproof of ω^{+} indexed by d, say h_d.
  3. (If necessary) normalize h_d.
- The following is the result:

$$
\dfrac{\begin{matrix}\vdots\ h_d\\ B(\alpha)\end{matrix}}{B[\alpha/D]}\ S
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Impredicative Normalization

Lemma

There is an operation $\mathscr{D}_{\mathbf{0}}$ *on terms such that if* $\vdash_{\mathbf{0}}^{\boldsymbol{a}} \boldsymbol{M} : \boldsymbol{A}$ *with* $\boldsymbol{A} \in \Pi_{\mathbf{1}}^{\mathbf{1}}$*, then* $\models^{\mathbf{D_0}\boldsymbol{a}} \mathscr{D}_{\mathbf{0}}(\boldsymbol{M}) : \boldsymbol{A}$.

Embedding Theorem

Theorem

Let $\boldsymbol{M}$ be a term such that all subterms of it have levels $\leq \boldsymbol{m}$. Also, let $\vec{\mathbf{y}}$ be any sequence of variables such that $\boldsymbol{M}\vec{\mathbf{y}}$ is well-typed. Then, there exists $\boldsymbol{k}$ such that $\vdash_{\boldsymbol{m}}^{\Omega_1 \times \boldsymbol{k}} \boldsymbol{M}\vec{\mathbf{y}}$.

Corollary

If $\boldsymbol{M} : \boldsymbol{A}$ and $\vec{\mathbf{y}}$ is any sequence of variables such that $\boldsymbol{A}$ is Π_1^1 and $\boldsymbol{M}\vec{\mathbf{y}}$ is well-typed, then $\models^{\mathbf{D_0 D_1^{\boldsymbol{m}}} \Omega \times \boldsymbol{n}} \mathscr{D}_0 \mathscr{D}_1^{\boldsymbol{m}}(\boldsymbol{M}\vec{\mathbf{y}}) : \boldsymbol{C}$ for some $\boldsymbol{n}, \boldsymbol{C}$.

Slow Growing Hierarchy on Ordinals

Next, we introduce the slow growing hierarchy by which we climb down the set of countable trees (T_1).

Definition

$G_a : \mathbb{N} \to \mathbb{N}$ for $a \in \mathrm{T}_1$ is defined by induction on a:

1. $G_0(n) := 0$,
2. $G_{a+1}(n) := G_a(n) + 1$,
3. $G_a(n) := G_{a[n]}(n)$ if $\mathrm{tp}(a) := \omega$. (when a is limit)

Note that $\omega[n] = n$ holds (by the definition).

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It holds that $G_{a+b}(n) = G_a(n) + G_b(n)$.

$$G_k(n) \quad = \quad k,$$

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$$\begin{aligned} G_k(n) &= k, \\ G_\omega(n) &= G_{\omega[n]}(n) = G_n(n) = n, \\ G_{D_0 1}(n) &= G_{\omega\times(n+1)}(n) = G_\omega(n) \times (n+1) = n \times (n+1). \end{aligned}$$

Recall that $G_a(4)$ is a natural number even if $a \in \mathrm{T}_1$ is infinite.

Lemma

If $\vdash_0^a \mathsf{S}^m 0$ *with* $4 \le a \in T_1$, *then* $\mathsf{S}^m 0 < G_a(4)$.

The Upper Bound Theorem

Definition

A function $\boldsymbol{f} : \mathbb{N} \to \mathbb{N}$ is *representable* in our system if there is a term $\boldsymbol{M} : N \Rightarrow N$ such that $\boldsymbol{M}(\mathsf{S}^n\mathbf{0}) \to_\beta^* \mathsf{S}^k\mathbf{0}$ iff $\boldsymbol{f}(\boldsymbol{n}) = \boldsymbol{k}$, where $\mathsf{S}^k\mathbf{0}$ is the Church numeral corresponding to $\boldsymbol{k}$.

Theorem

Let $\boldsymbol{f}$ be a representable function in our system with $\boldsymbol{M} : \boldsymbol{N} \Rightarrow \boldsymbol{N}$. Then, $\models_{\mathbf{0}}^{\mathbf{D_0}(\boldsymbol{d}\times(\boldsymbol{n}+\mathbf{1}))} \mathscr{D}_{\mathbf{0}}\mathscr{D}_{\mathbf{1}}^{\boldsymbol{m}}(\boldsymbol{M}\mathsf{S}^{\boldsymbol{n}}) : \boldsymbol{N}$ with $\boldsymbol{d} = \mathbf{D}_{\mathbf{1}}^{\boldsymbol{m}}(\Omega \times \boldsymbol{m})$ for some $\boldsymbol{m}$. Therefore, there is $\boldsymbol{m}$ such that for all $\boldsymbol{n} \geq \boldsymbol{m}$

$$|\mathscr{D}_{\mathbf{0}}\mathscr{D}_{\mathbf{1}}^{\boldsymbol{m}}(\boldsymbol{M}\mathsf{S}^{\boldsymbol{n}})| < \boldsymbol{G}_{\mathbf{D_0}\mathbf{D}_{\mathbf{1}}^{\boldsymbol{m}+\mathbf{2}}\mathbf{0}}(\boldsymbol{n}).$$

Lower Bound Theorem

Theorem (Schwichtenberg90)

For any $\boldsymbol{m}$, *we can formally prove in arithmetic:* $\forall x \exists y (\mathbf{D_0 D_1^m 0})[x]^y = \mathbf{0}$.

Theorem (Aehlig05, 08)

1. *The following are equivalent:*
   - 1.1 $ID_0^c \vdash \forall x \exists y R(x,y)$,
   - 1.2 $HA_1^2 \vdash \forall x.\mathbb{N} \to \neg \forall y(\mathbb{N}y \to \neg R(x,y))$.
2. *If* $HA_1^2 \vdash \forall x.\mathbb{N} \to \neg \forall y(\mathbb{N}y \to \neg R(x,y))$, *then there is a term in our system computing this function on Church numerals, that is, for every* $\boldsymbol{n}$ *the term* tc_n *reduced to a Church numeral* c_l *and* $R(n,l)$ *holds.*

Theorem

The function expressed by $\forall x \exists y (\mathbf{D_0 D_1^m 0})[x]^y = \mathbf{0}$ *is representable in our system.*

Summary

- As expected, the complexity of a term of type $\boldsymbol{N} \Rightarrow \boldsymbol{N}$ is bounded:

$$|\mathscr{D}_{\mathbf{0}}\mathscr{D}_{\mathbf{1}}^{\boldsymbol{m}}(\boldsymbol{M}\mathbf{S}^{\boldsymbol{n}})| < \boldsymbol{G}_{\mathbf{D_0 D_1^{\mathit{m}+2} 0}}(\boldsymbol{n}).$$

- We can iterate our approach to handle with Brouwer's ordinals (and more...)

$$\boldsymbol{O} : \forall\alpha.((\boldsymbol{N} \Rightarrow \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha)$$

- If we formalized tree ordinals and the slow growing hierarchy, then we could see the bound in a *visible* way by considering concrete examples.
- We computed the upperbound of the length of β-reductions in our system, too.

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