# Horn Fragments of Halpern-Shoham Interval Temporal Logic

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Joint work with

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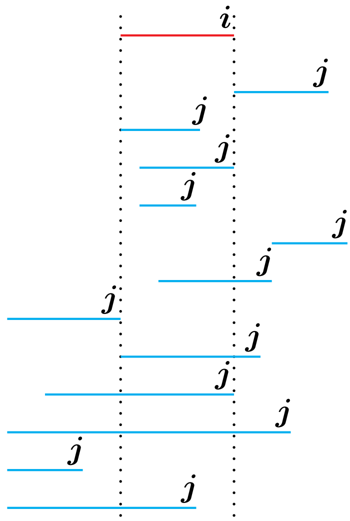
# Allen's interval relations

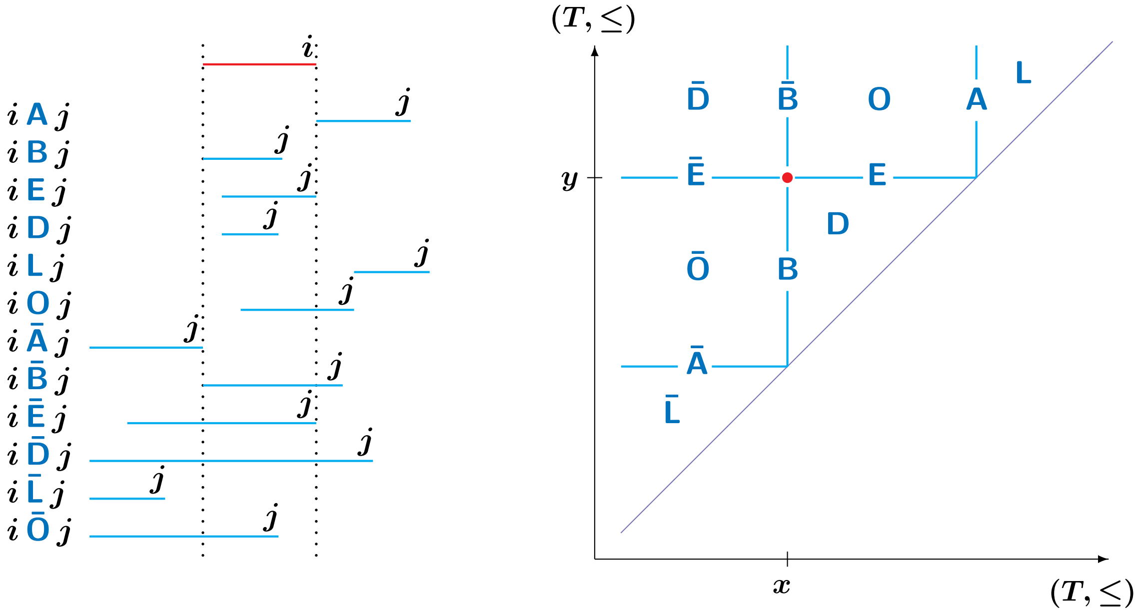
**Time:** linear order $(T, \leq)$

**Intervals:** $i = \langle x, y \rangle$ with $x \leq y$




- $i$ A $j$ — After
- $i$ B $j$ — Begins
- $i$ E $j$ — Ends
- $i$ D $j$ — During
- $i$ L $j$ — Later
- $i$ O $j$ — Overlaps
- $i$ $\bar{\text{A}}$ $j$
- $i$ $\bar{\text{B}}$ $j$
- $i$ $\bar{\text{E}}$ $j$
- $i$ $\bar{\text{D}}$ $j$
- $i$ $\bar{\text{L}}$ $j$
- $i$ $\bar{\text{O}}$ $j$

# Allen's interval relations – 2D representation




$\bar{\mathsf{B}}\mathsf{E}$-fragment = 'expanding' modal product $(T,\leq)\times^{exp}(T,\leq)$

# Halpern-Shoham interval temporal logic

$\mathcal{HS}$: propositional multi-modal logic over the 12 Allen relations

**formulas:** $\varphi \ ::= \ \top \mid \bot \mid p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \langle \mathsf{R} \rangle \varphi \mid [\mathsf{R}]\varphi$

$p \in$ *Variables*, $\mathsf{R} \in \{\mathsf{L}, \mathsf{B}, \mathsf{A}, \mathsf{D}, \mathsf{O}, \mathsf{E}, \bar{\mathsf{L}}, \bar{\mathsf{B}}, \bar{\mathsf{A}}, \bar{\mathsf{D}}, \bar{\mathsf{O}}, \bar{\mathsf{E}}\}$

**models:** $\mathfrak{M} = (T, \leq, \nu)$ where $\nu : \mathit{Intervals} \to 2^{\mathit{Variables}}$

**truth-relation:**

$\mathfrak{M}, i \models \langle \mathsf{R} \rangle \varphi \iff \mathfrak{M}, j \models \varphi$ for **some** interval $j$ with $i\mathsf{R}j$

$\mathfrak{M}, i \models [\mathsf{R}]\varphi \iff \mathfrak{M}, j \models \varphi$ for **all** intervals $j$ with $i\mathsf{R}j$

**Variants:**

- discrete, dense, finite, ... linear orders
- open, closed, semi-closed intervals
- 'punctual' $\langle x, x \rangle$ intervals allowed/disallowed
- 'reflexive/irreflexive' interpretation of the Allen relations

**E.g.:** $\langle x_1, y_1 \rangle \mathsf{B} \langle x_2, y_2 \rangle \iff x_1 = x_2$ and $y_2 \leq y_1$

$\langle x_1, y_1 \rangle \mathsf{B} \langle x_2, y_2 \rangle \iff x_1 = x_2$ and $y_2 < y_1$

$p \wedge [\mathsf{R}]\neg p$ **is not satisfiable with reflexive semantics**

# The satisfiability problem of $\mathcal{HS}$

**Task:** Given a formula $\varphi$ and a class $\mathcal{C}$ of linear orders, are there some model $\mathfrak{M}$ based on a timeline from $\mathcal{C}$ and interval $i$ such that $\mathfrak{M}, i \models \varphi$ ?

**Undecidable** over any 'unbounded' class of linear orders

*Halpern–Shoham 1991*

**'Taming' attempts so far:**

- constraining the underlying linear orders
- relativisations
- softening the semantics
- restricting the set of available modal operators
- coarsening the interval relations
- restricting the nesting of modal operators

# Horn fragments of $\mathcal{HS}$

**clausal normal form** of $\mathcal{HS}$-formulas:

$$\varphi \quad ::= \quad \boldsymbol{\lambda} \mid \neg\boldsymbol{\lambda} \mid \forall(\boldsymbol{\lambda}_0 \wedge \dots \wedge \boldsymbol{\lambda}_n \to \boldsymbol{\lambda}_{n+1} \vee \dots \vee \boldsymbol{\lambda}_{n+m}) \mid \varphi_1 \wedge \varphi_2$$

where $\forall\psi = \bigwedge_{\mathsf{R}}(\psi \wedge [\mathsf{R}]\psi \wedge [\bar{\mathsf{R}}]\psi)$

$$\boldsymbol{\lambda} \quad ::= \quad \top \mid \bot \mid \boldsymbol{p} \mid \langle \mathsf{R}\rangle\boldsymbol{\lambda} \mid [\mathsf{R}]\boldsymbol{\lambda}$$

$\mathcal{HS}_{horn}$ : $\quad \varphi \quad ::= \quad \boldsymbol{\lambda} \mid \forall(\boldsymbol{\lambda}_0 \wedge \dots \wedge \boldsymbol{\lambda}_n \to \boldsymbol{\lambda}) \mid \varphi_1 \wedge \varphi_2$

$\mathcal{HS}_{core}$ : $\quad \varphi \quad ::= \quad \boldsymbol{\lambda} \mid \forall(\boldsymbol{\lambda}_0 \to \boldsymbol{\lambda}_1) \mid \forall(\boldsymbol{\lambda}_0 \wedge \boldsymbol{\lambda}_1 \to \bot) \mid \varphi_1 \wedge \varphi_2$

$\mathcal{HS}^{\Box}_{horn}$ and $\mathcal{HS}^{\Box}_{core}$ : NO diamond operators $\langle \mathsf{R}\rangle$ in $\boldsymbol{\lambda}$

# What can we express?

**Some syntactic sugar:** for $\psi = \boldsymbol{\lambda}_1 \wedge \cdots \wedge \boldsymbol{\lambda}_k$

- $\forall(\psi \to \neg\boldsymbol{\lambda}) \quad \equiv \quad \forall(\psi \wedge \boldsymbol{\lambda} \to \bot)$
- $\forall\big(\psi \to \boldsymbol{\lambda}'_1 \wedge \cdots \wedge \boldsymbol{\lambda}'_n\big) \quad \equiv \quad \bigwedge_{i=1}^{n} \forall\big(\psi \to \boldsymbol{\lambda}'_i\big)$
- $\forall\big(\psi \to [\mathsf{R}](\boldsymbol{\lambda}'_1 \wedge \cdots \wedge \boldsymbol{\lambda}'_n \to \boldsymbol{\lambda})\big) \quad \equiv \quad \forall(\psi \to [\mathsf{R}]\boldsymbol{p}) \wedge \forall(\boldsymbol{p} \wedge \boldsymbol{\lambda}'_1 \wedge \cdots \wedge \boldsymbol{\lambda}'_n \to \boldsymbol{\lambda})$
- $\forall\big(\psi \to \langle\mathsf{R}\rangle(\boldsymbol{\lambda}'_1 \wedge \cdots \wedge \boldsymbol{\lambda}'_n \to \boldsymbol{\lambda})\big) \quad \equiv \quad \forall(\psi \to \langle\mathsf{R}\rangle\boldsymbol{p}) \wedge \forall(\boldsymbol{p} \wedge \boldsymbol{\lambda}'_1 \wedge \cdots \wedge \boldsymbol{\lambda}'_n \to \boldsymbol{\lambda})$
- $\forall(\langle\mathsf{R}\rangle\boldsymbol{\lambda} \wedge \psi \to \boldsymbol{\lambda}') \quad \equiv \quad \forall(\boldsymbol{\lambda} \to [\bar{\mathsf{R}}](\psi \to \boldsymbol{\lambda}'))$

**Advanced courses cannot be given during the morning sessions.**

$\forall(\langle\bar{\mathsf{D}}\rangle \mathit{MorningSession} \wedge \mathit{AdvancedCourse} \to \bot)$

$\forall(\langle\bar{\mathsf{B}}\rangle \mathit{LectureDay} \wedge \langle\mathsf{A}\rangle \mathit{Lunch} \leftrightarrow \mathit{MorningSession})$

**Teaching is both downward and upward hereditary.**

$\forall(\mathit{teaching} \to [\mathsf{D}]\mathit{teaching})$

$\forall(([\mathsf{D}]\langle\mathsf{O}\rangle \mathit{teaching} \vee \langle\bar{\mathsf{D}}\rangle \mathit{teaching}) \wedge \langle\mathsf{B}\rangle \mathit{teaching} \wedge \langle\mathsf{E}\rangle \mathit{teaching} \to \mathit{teaching})$

# Our results on the satisfiability problem

| | irreflexive semantics | reflexive semantics |
|---|---|---|
| $\mathcal{HS}_{horn}$ | undecidable* | |
| $\mathcal{HS}_{core}$ | undecidable* | PSPACE-hard*<br>decidable? |
| $\mathcal{HS}^{\Box}_{horn}$ | **discrete:** undecidable<br>**dense:** PTIME-complete | PTIME-complete** |
| $\mathcal{HS}^{\Box}_{core}$ | **discrete:** PSPACE-hard<br>decidable?<br>**dense:** in PTIME | in PTIME** |

*over **any** 'unbounded' class of linear orders

**over **any** nonempty class of linear orders

<u>Note:</u> **propositional Datalog is PTIME-complete**

**its core fragment is NLOGSPACE-complete**

# Proving PSPACE lower bounds

In both

- $\mathcal{HS}^{\Box}_{core}$ over **discrete** linear orders with **irreflexive** semantics
- $\mathcal{HS}_{core}$ over arbitrary linear orders with arbitrary semantics

it is possible

- to identify/generate an infinite (or unbounded finite) sequence of 'units'
- to pass polynomial-size information from one unit to the next

$\rightsquigarrow$ **PSPACE-hardness**

- ( $\mathcal{HS}^{\Box}_{core}$ over **dense** linear orders OR with **reflexive** semantics is **in PTIME**)
- ( $\mathcal{HS}_{core}$ over arbitrary linear orders with **irreflexive** semantics is even **undecidable**)

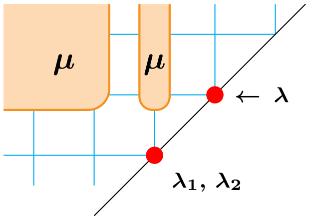
# PSpace lower bounds – the tricks

- in $\mathcal{HS}^{\Box}_{\text{core}}$ over **discrete** linear orders with **irreflexive** semantics:

  identifying units: $\forall(\textbf{unit} \leftrightarrow [\mathsf{E}]\,\bot)$

  passing poly-size info to next unit: $[\lambda_1 \& \lambda_2 \Rightarrow \lambda]$

  $$\begin{array}{l}\forall(\boldsymbol{\lambda}_1 \to [\mathsf{A}]\,\boldsymbol{\mu})\\ \forall(\boldsymbol{\lambda}_2 \to [\mathsf{A}]\,[\bar{\mathsf{E}}]\,\boldsymbol{\mu})\\ \forall([\mathsf{A}]\,[\bar{\mathsf{A}}]\,\boldsymbol{\mu} \to \boldsymbol{\lambda})\end{array}$$



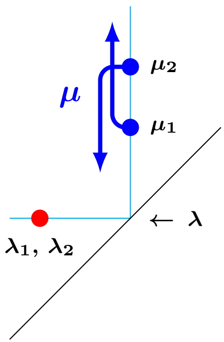

- in $\mathcal{HS}_{\text{core}}$ over arbitrary linear orders with arbitrary semantics:

  generating units:

  $$\begin{array}{l}\textbf{unit} \land \forall(\textbf{unit} \to \langle\mathsf{A}\rangle\textbf{unit})\\ \forall(\textbf{unit} \to \neg[\mathsf{D}]\textbf{unit})\end{array}$$

  'compressing' info to be passed: $[\lambda_1 \& \lambda_2 \Rightarrow \lambda]$

  $$\begin{array}{l}\forall(\boldsymbol{\lambda}_1 \to \langle\mathsf{A}\rangle\boldsymbol{\mu}_1) \land \forall(\boldsymbol{\lambda}_2 \to \langle\mathsf{A}\rangle\boldsymbol{\mu}_2)\\ \forall(\boldsymbol{\mu}_2 \to \neg\langle\bar{\mathsf{B}}\rangle\boldsymbol{\mu}_1)\\ \forall(\boldsymbol{\mu}_1 \to \boldsymbol{\mu} \land [\bar{\mathsf{B}}]\boldsymbol{\mu}) \land \forall(\boldsymbol{\mu}_2 \to [\mathsf{B}]\boldsymbol{\mu})\\ \forall([\mathsf{A}]\,\boldsymbol{\mu} \to \boldsymbol{\lambda})\end{array}$$

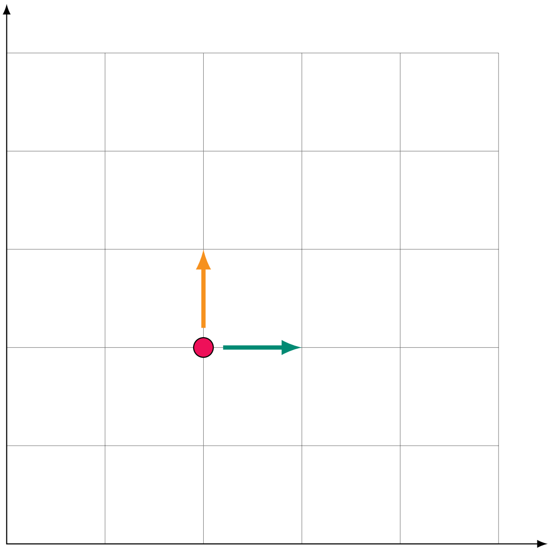


# Proving undecidability: encoding 'grid-based' problems

- generating the $\omega \times \omega$-grid (or arbitrarily large finite 'initial segment' of it)
- passing poly-size info to right- and up-neighbours

$\rightsquigarrow$ **encoding**

- tilings
- Turing machine computations
- counter machine computations
- Post correspondence problem
- ...



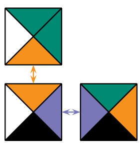
# Often 'half-grid' is enough

$$nw_{\omega\times\omega} = \{(i,j) : i \leq j < \omega\}$$

the $nw_{\omega\times\omega}$ **tiling problem:**

given a finite set $T$ of tile types $t = (\textit{left}(t), \textit{right}(t), \textit{up}(t), \textit{down}(t))$

decide whether there exists $\tau : nw_{\omega\times\omega} \to T$ such that, for all $i \leq j < \omega$,

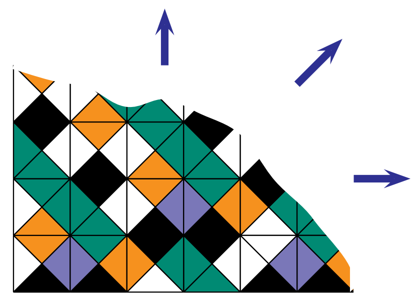


$$\textit{up}(\tau(i,j)) = \textit{down}(\tau(i,j+1))$$

and

$$\textit{left}(\tau(i,j)) = \textit{right}(\tau(i+1,j)) \quad \text{whenever } i < j$$

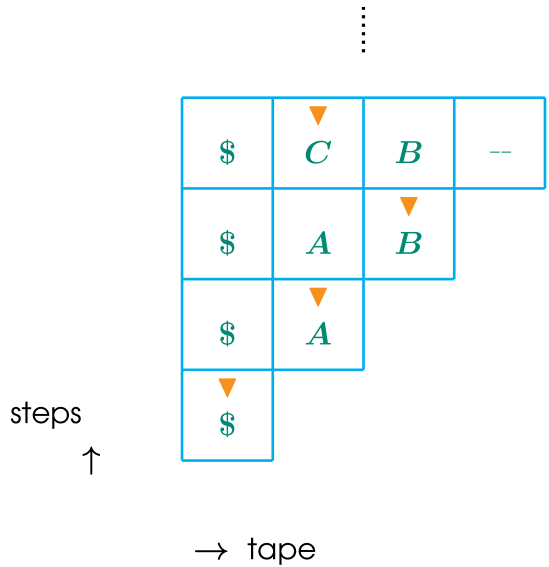
**(Berger 1966):** the $\omega \times \omega$ tiling problem is **undecidable**



$\rightsquigarrow$ the $nw_{\omega\times\omega}$ tiling problem is **undecidable**

# Often ‘half-grid’ is enough

**Turing machine** computations starting with **empty tape** on $nw_{\omega\times\omega}$ :



# Diagonal encoding of $nw_{\omega\times\omega}$

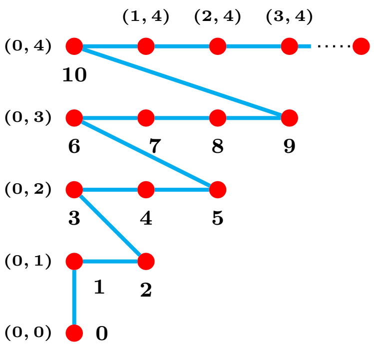
$\mathcal{HS}$-models are kind of **grid-like**

**BUT** $\mathcal{HS}$ has **NO** 'right-neighbour' and 'up-neighbor' operators

not even over discrete orders with irreflexive semantics

**Idea:**

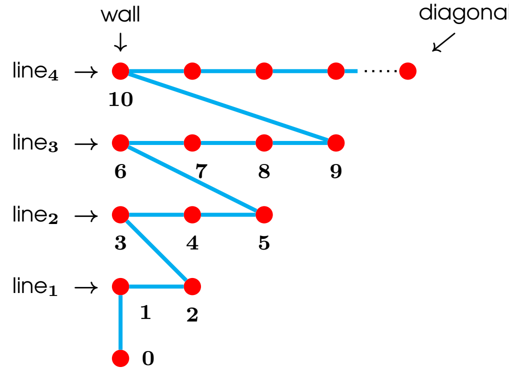
*Harel 1985, Halpern–Shoham 1991, Marx–Reynolds 1999, Reynolds–Zakharyaschev 2001*




- generate an infinite **sequence** of 'units'
- **right-neighbour** of each non-diagonal unit **is the next** unit in the sequence
- use **'local' pointers** to access the units representing the **up-neighbour** of each grid-location

# Diagonal encoding of $nw_{\omega\times\omega}$ – version 1

*Halpern–Shoham 1991*

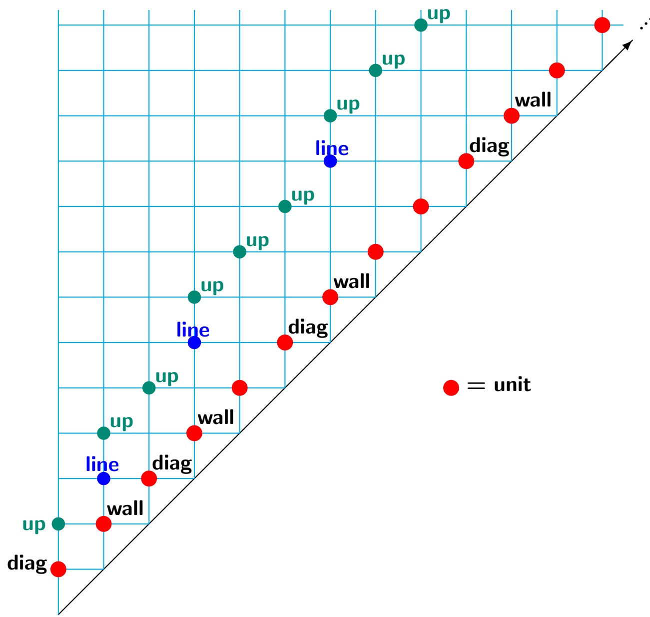



- start of **line**$_1$ is $1$, and **up_of**$(0) = 1$
- start of **line**$_{i+1}$ is the end of **line**$_i + 1$, for all $i > 0$
- every line starts with some $n$ on the wall and

  ends with some $m$ on the diagonal
- if $n$ is in **line**$_i$ then **up_of**$(n)$ is in **line**$_{i+1}$
- if $m < n$ then **up_of**$(m) <$ **up_of**$(n)$
- if $n > 0$ is on the wall then there is $m$ with **up_of**$(m) = n$
- if $n$ is neither on the wall nor on the diagonal

  then there is $m$ with **up_of**$(m) = n$

# Version 1 is doable in $\mathcal{HS}_{core}$ with irreflexive semantics




FOR EXAMPLE:

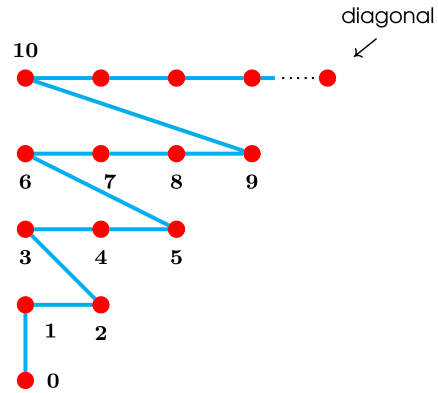
- if $n$ is neither on the wall nor on the diagonal then there is $m$ with $\textbf{up\_of}(m) = n$

  $[\langle \bar{\mathsf{D}} \rangle \textbf{line} \,\&\, \textbf{unit} \Rightarrow \langle \mathsf{A} \rangle \langle \bar{\mathsf{A}} \rangle \textbf{up}]$

# Diagonal encoding of $nw_{\omega\times\omega}$ – version 2

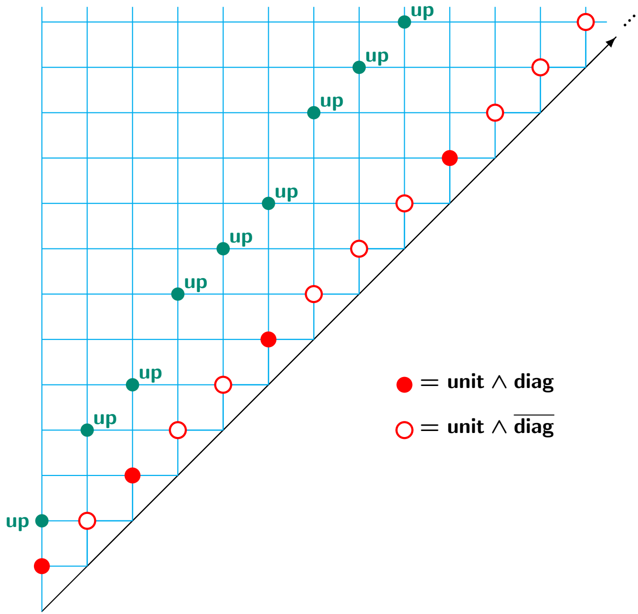
*Marx–Reynolds 1999*

*Reynolds–Zakharyaschev 2001*




- $0$ is on the diagonal, and $\textbf{up\_of}(0) = 1$
- if $n$ is on the diagonal, then $\textbf{up\_of}(n) + 1$ is on the diagonal
- if $n$ is the up-neighbour of some location, then $n$ is not on the diagonal
- if $n$ is not on the diagonal, then $\textbf{up\_of}(n + 1) = \textbf{up\_of}(n) + 1$
- if $n$ is on the diagonal, then $\textbf{up\_of}(n + 1) = \textbf{up\_of}(n) + 2$

# Version 2 is doable in $\mathcal{HS}_{horn}$




FOR EXAMPLE:

- if $n$ is the up-neighbour of some location, then $n$ is not on the diagonal

$$\forall\big(\mathbf{up} \to [E]\ (\mathbf{unit} \to \overline{\mathbf{diag}})\big) \qquad \wedge \qquad \forall(\mathbf{diag} \wedge \overline{\mathbf{diag}} \to \bot)$$

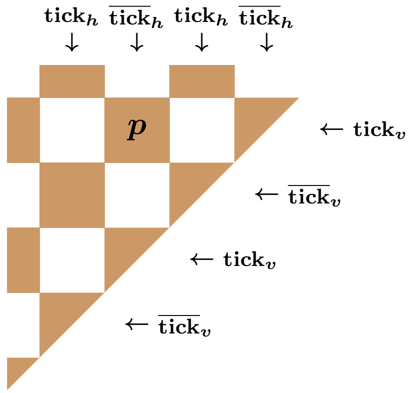
# Chessboard – a trick for reflexive semantics/arbitrary orders

With these choices it is not possible to generate a **unique** unit-sequence,

BUT **'discretising'** $\mathcal{HS}$-models can be done in $\mathcal{HS}_{\text{horn}}$ :

*Spaan 1993*

*Artale–Kontchakov–Lutz–Wolter–Zakharyaschev 2007*




- using horizontal and vertical 'tick'-variables
- forcing the 'stability' of relevant variables in each chessboard square
- combining chessboards with the previous techniques

⇝ $\mathcal{HS}_{\text{horn}}$ **is undecidable** over **arbitrary** linear orders with **arbitrary** semantics

## Some more open questions on the satisfiability problem

- **mixing Horn fragments with other 'taming' techniques**
- *Halpern–Shoham 1991:* more general temporal structures

  - $(T, \leq)$ is a partial order
  - for every $t \leq t'$, the interval $[t, t']$ is linearly ordered

  $\rightsquigarrow$ $(T, \leq)$ **is forest-like**

  **What about Horn fragments of $\mathcal{HS}$ over these?**

  *Gabelaia–K–Wolter–Zakharyaschev 2005:*

  bimodal products of transitive structures are undecidable

- **Can we make use of the grid-like nature of $\mathcal{HS}$-models directly,**

  **without diagonally encoding grids?**

  *Hampson-K 2015:* bimodal product logics

  'difference operator' $\times$ 'linear order' are undecidable

  proof is by encoding **counter machine** computations

  **directly on the grid-like** models