Many-valued modal logics: axiomatizability issues

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- Many normal (classical) modal logics: finite model property + finite axiomatizability $\Rightarrow$ decidability

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- Applications in industry/AI etc. + (classical) mathematical interest for its relation with Universal Algebra and more specific algebraic fields (eg. lattice-ordered abelian groups with unit).
- Many well-known infinitely-valued cases still decidable (Ł, Gödel, Product, H-BL...)

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- syntactically/algebraically not so clear what does this mean in the basic case
- usual approach: valuation of Kripke models: both the accessibility relation ($\rightarrow$ fuzzy relation) and the formulas at each world
- systematic study is under development:
  - Some modal MV logics have been axiomatised, but most have not.*
  - In many (most) cases, there is no (usual) finite model property or (known) R.E/finite axiomatization $\Rightarrow$ decidability??

* if we knew more on the axiomatizations $\Longrightarrow$ algebraic semantics, decidability/complexity, duality, etc.

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- Product modal logics either -only their infinitary correspondent, and adding rational constants (Vidal, 2017).
- can we say something else??

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The non-modal part

Definition

A FL_{ew} algebra $\mathbf{A}$ is $\langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee\rangle$ is a lattice,
- $\langle A, \odot, 1\rangle$ is a commutative monoid
- $x \odot y \leq z \iff x \leq y \rightarrow z$ (residuation law)
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well known cases

* $[\mathbf{0}, \mathbf{1}]_{\text{Ł}}$ (where $x \odot y = \max\{0, x + y - 1\}$), $[\mathbf{0}, \mathbf{1}]_{\Pi}$ (where $\odot = \cdot$), $[\mathbf{0}, \mathbf{1}]_{\mathbf{G}}$ (where $\odot = \wedge$).
* 1-generated subalgebras of $[0, 1]_{\Pi}$: $\{0, 1\} \cup \{a^k \colon k \in \omega\}$ for some $a \in (0, 1)$.

MV-Kripke models

Language: $\odot, \rightarrow, \wedge, 0$ plus two unary (modal) symbols ($\Box, \Diamond$).

Definition

A (crisp) **A**-*Kripke model* $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ where:

- $R \subseteq W \times W$ (Rus stands for $\langle u, s\rangle \in R$)
- $e : W \times Var \rightarrow A$ uniquely extended by:
  - $e(u, \varphi \star \psi) = e(u, \varphi) \star e(u, \psi)$, for $\star \in \{\odot, \wedge, \rightarrow\}$,
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- $\mathcal{M}_{\mathcal{C}}$ = class of safe Kripke models over algebras in $\mathcal{C}$, and $\omega\mathcal{M}_{\mathcal{C}}$ the finite ones.
- $\mathcal{M}_{Ł}$, $\mathcal{M}_{\Pi}$, $\mathcal{M}_{\Pi_1}$ = classes of Kripke models valued respectively over $[0,1]_{Ł}$, $[0,1]_{\Pi}$ and 1-gen. subalgebras of $[0,1]_{\Pi}$.

Modal logics over residuated lattices

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- **(Local modal logic)**: $\Gamma \vdash^{l}_{\mathcal{K}} \varphi$ iff for all $\mathfrak{M} \in \mathcal{K}$ and for all $u \in W$,

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Theorem

The following are undecidable:

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Some general ideas of the proof...

- Post correspondence problem: given $\langle v_1, w_1\rangle, \ldots, \langle v_n, w_n\rangle$ of pairs of numbers in some base $s > 1$, it is **undecidable** whether there exist $i_1, \ldots, i_k$ with $i_j \in \{1, \ldots, n\}$ such that $v_{i_1} \cdots v_{i_k} = w_{i_1} \cdots w_{i_k}$.

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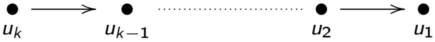
Similar for the local transitive case.

- The $\Rightarrow$ direction exploits non-contractivity of some algebra in the class.

...Some general ideas of the proof

- The $\Leftarrow$ direction uses weakly saturation and non-contractivity to prove that if $\Gamma_P \not\Vdash_{\mathcal{K}} \varphi_P$ then it happens in a model with structure



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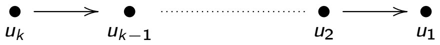
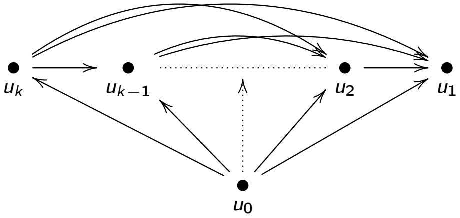


  with an evaluation that is then easily turnable into a solution of P.
- For the local case, the approach is very similar, getting a model of the form:



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Lemma

If $\mathcal{C}$ of R.L is as in Lemma $(\star)$ and $\models_{\mathcal{C}}$ is decidable, then $\vdash^{g}_{\omega\mathcal{M}_{\mathcal{C}}}$ is not R.E, and so, not axiomatizable.

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However, it is not the case that $\vdash^{g}_{\omega\mathcal{M}_{\text{Ł}}} = \vdash^{g}_{\mathcal{M}_{\text{Ł}}}$, nor for the product case ...

so what about $\vdash^{g}_{\mathcal{M}_{\text{Ł}}}$ and $\vdash^{g}_{\mathcal{M}_{\Pi}}$? (the modal Łukasiewicz/product logics?)

The Łukasiewicz case

A model $\mathfrak{M}$ is witnessed iff for all $v \in W$, φ, there are $w_{\Box\varphi}$, $w_{\Diamond\varphi}$

$$e(v, \Box\varphi) = e(w_{\Box\varphi}, \varphi) \quad \text{and} \quad e(v, \Diamond\varphi) = e(w_{\Diamond\varphi}, \varphi)$$

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We have completeness wrt. finite-width models... but the depth might still be infinite

The Łukasiewicz case

Lemma

$\Gamma \vdash^{g}_{\omega\mathcal{M}_{Ł}} \varphi$ iff $\Gamma, \Upsilon(p,q) \vdash^{g}_{\mathcal{M}_{Ł}} \varphi \vee \Psi(p,q)$ for any $p, q \notin \mathcal{V}ars(\Gamma, \varphi)$ and

- $\Upsilon(p,q) \coloneqq \{\Box 0 \vee (p \leftrightarrow \Box p), \Box 0 \vee (\Box p \leftrightarrow \Diamond p), (q \leftrightarrow p \odot \Box q)\}$
- $\Psi(p,q) \coloneqq p \vee \neg p \vee q \vee \neg q$.

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$\Gamma \vdash^{g}_{\omega\mathcal{M}_Ł} \varphi$ iff $\Gamma, \Upsilon(p,q) \vdash^{g}_{\mathcal{M}_Ł} \varphi \vee \Psi(p,q)$ for any $p, q \notin \mathcal{V}ars(\Gamma, \varphi)$ and

- $\Upsilon(p,q) \coloneqq \{\Box 0 \vee (p \leftrightarrow \Box p), \Box 0 \vee (\Box p \leftrightarrow \Diamond p), (q \leftrightarrow p \odot \Box q)\}$
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Given a finite set of formulas Γ, φ, whether $\Gamma \equiv \Gamma_0 \cup \Upsilon(p,q)$ and $\varphi \equiv \varphi_0 \vee \Psi(p,q)$ is a decidable process.

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Theorem

The finitary companion of the modal Lukasiewicz logic is not RE, and so, is not axiomatizable.

The Product case

! not known anything like the completeness of $\vdash^{g}_{\mathcal{M}_{\Pi}}$ wrt witnessed models (only a partial result, not generalizable, for theorems).

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$\Gamma \vdash_{\omega\mathcal{M}_{\Pi_1}} \varphi$ iff $\Gamma, \Upsilon(p,q), QW(\Gamma,\varphi) \vdash_{\mathcal{M}_{\Pi_1}} \varphi \vee \Psi(p,q)$ for $p, q, \Upsilon(p,q), \Psi(p,q)$ as in the Ł case and

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Corollary

The finitary companion of $\vdash_{\mathcal{M}_{\Pi_1}}$ is not RE, and so, not axiomatizable.

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Given Γ, φ, there is a set of variables $\mathcal{V}'$ defined from $\mathcal{V}ar(\Gamma, \varphi)$ and two sets of formulas $\Sigma(\Gamma, \varphi, \mathcal{V}')$, $\Theta(\varphi, \mathcal{V}')$ such that $\Gamma \vdash_{\mathcal{M}_{\Pi_1}} \varphi$ iff $\Sigma(\Gamma, \varphi, \mathcal{V}') \vdash_{\mathcal{M}_{\Pi}} \Theta(\varphi, \mathcal{V}')$.

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Grazie mille!