Łukasiewicz logic: 98 years

in memoriam Roberto Cignoli

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THEOREM (D.M, **2018**) *Let* $\rhd$ *be a continuous* $[0,1]$-*valued map defined on the real unit square* $[0,1]^2$ *such that*

$$x \rhd y = 1 \ \textit{ iff } \ x \leq y, \ \textit{and}$$

$$(x \rhd (y \rhd z)) = (y \rhd (x \rhd z)).$$

Set $\neg x = x \rhd 0$ *and* $x \oplus y = (x \rhd 0) \rhd y$.

Then the algebra $([0,1], 0, \neg, \oplus)$ *satisfies the equations:*

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$$x \oplus 0 = x$$

$$\neg\neg x = x$$

$$x \oplus \neg 0 = \neg 0$$

$$y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$$

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$$y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$$

These equations define MV-algebras, (Chang, TAMS, 1958), the algebras of Łukasiewicz ∞-valued logic (**1920**)

explaining the Łukasiewicz axiom $((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$

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Let the map $\triangleright : [0,1]^2 \rightarrow [0,1]$ remotely sounds as “implication”,

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If the axiom $((x \rhd y) \rhd y) \rhd ((y \rhd x) \rhd x)$ fails then $\rhd$ is not continuous

C*-algebras and AF C*-algebras

- An algebra A over $\mathbf{C}$ is called a *$*$-algebra* iff A is equipped with a linear map $* : A \to A$ such that for all $y, z \in A$ and $\mu \in \mathbf{C}$, $z^{**} = z$, $(zy)^* = y^* z^*$ and $(\mu z)^* = \bar{\mu} z^*$.

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DIXMIER, GLIMM, BRATTELI
(quantum thermodynamic limits)

Elliott **partial** monoid

Projections p and q in an AF-algebra A are ***Murray-von Neumann equivalent*** if $p=x^*x$ and $q=xx^*$ for some element x of A.

L(A) = set of equivalence classes $[p]$ of projections p of A.

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THEOREM (Elliott's classification, J.Algebra 1976)

Two AF-algebras are isomorphic iff so are their Elliott partial monoids.

Elliott **total** monoid

THEOREM (G. Panti, D.M., J. Functional Analysis, 1993, building on Elliott's classification and on D.M., J. Functional Analysis 65, 1986)

Elliott's partial addition $+$ *has at most one extension to an associative, commutative, monotone operation* $\boldsymbol{+}$ *over* $\mathrm{L}(A)$ *such that* $[1-p]$ *is the smallest class* $[q]$, *satisfying* $[q] \boldsymbol{+} [p] = 1$.

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Such extension $+$ *exists if and only if* $\mathrm{L}(A)$ *is lattice-ordered, for short, A is an* **AF*l*-algebra**.

We write $\neg[p]$ *for* $[1-p]$. *The map* $A \to (\mathrm{L}(A), 0, \neg, +)$ *sends* ***AFl-algebras one-one onto countable MV-algebras.***

two functors behind this correspondence

THEOREM 1 (Elliott, Handelman, Goodearl, Effros, etc.)
Grothendieck K_0 *functor sends AF*l*-algebras one one onto countable lattice ordered abelian groups equipped with a distinguished order-unit (for short,* unital l-groups*)*

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Further developments by D.M. in Advances in Mathematics, 2018

Most AF algebras in the literature are AF*ℓ*

COUNTABLE MV ALGEBRA	*ITS AF C*-CORRESPONDENT*
$\{0,1\}$	$\mathbf{C}$
Łukasiewicz chain $\{0, 1/n, ..., (n-1)/n, 1\}$	$M_n(\mathbf{C})$, the $n \times n$ complex matrices
finite	finite-dimensional
boolean	commutative
atomless boolean	$C(2^{\omega})$
subalgebra of $\mathbf{Q} \cap [0,1]$	UHF, uniformly hyperfinite
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Chang algebra C	Behncke-Leptin $A_{0,1}$
totally ordered	with Murray-von Neumann comparability
free with one generator	the Farey algebra M_1
free with $\aleph_0$ generators	the universal AF C*-algebra M

First Application:

Free AF-algebras from free MV-algebras

first main effect of this correspondence

MV-algebras are specified by **equations.**

So **free** MV-algebras exist.

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Free MV-algebras yield genuinely **"free" AF*l*-algebras.**

free MV-algebras

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The variety of MV-algebras is generated by the unit real interval [0,1] *equipped with negation* $\neg x = 1 - x$ *and truncated sum* $\min(1, x+y)$

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Free n-generator MV-algebras consist of ***some*** *piecewise linear continuous* [0,1]*-valued functions with integer coefficients, defined over the cube* $[0,1]^n$.

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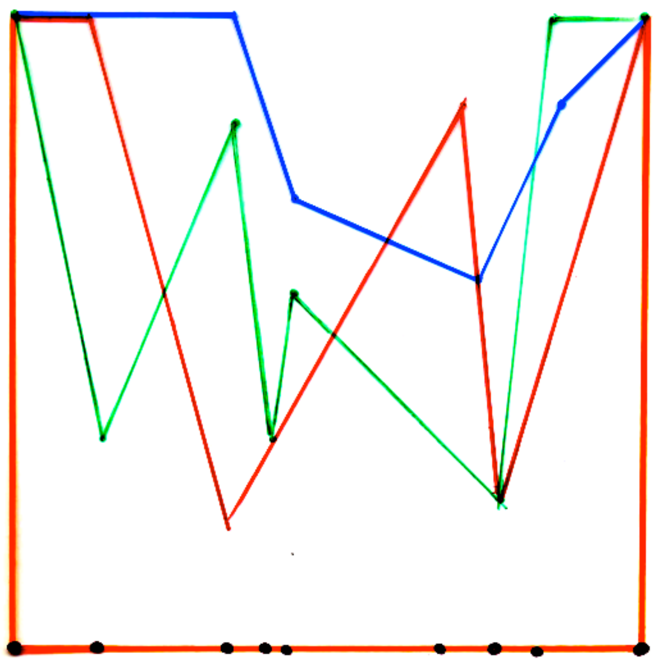
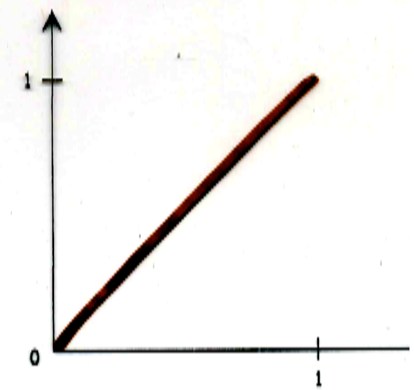
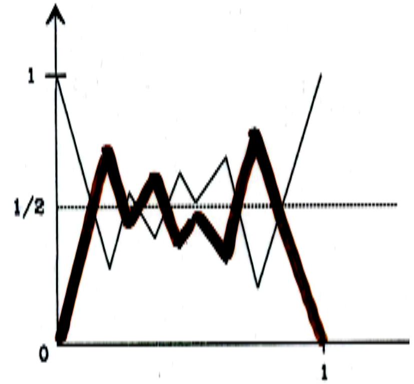
THEOREM (R.McNaughton, J. Symbolic Logic 1951)
Free n-generator MV-algebras consist of ***all*** *such functions*.

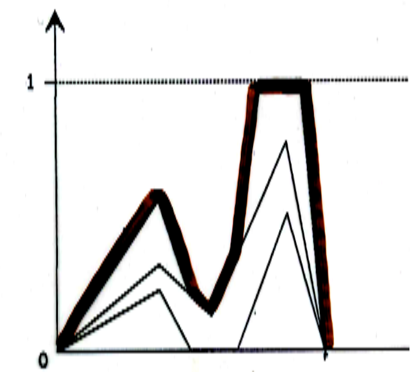
Example: the free one-generator MV-algebra F_1

F_1 consists of all piecewise linear continuous [0,1]*-valued functions with integer coefficients, defined over* [0,1].



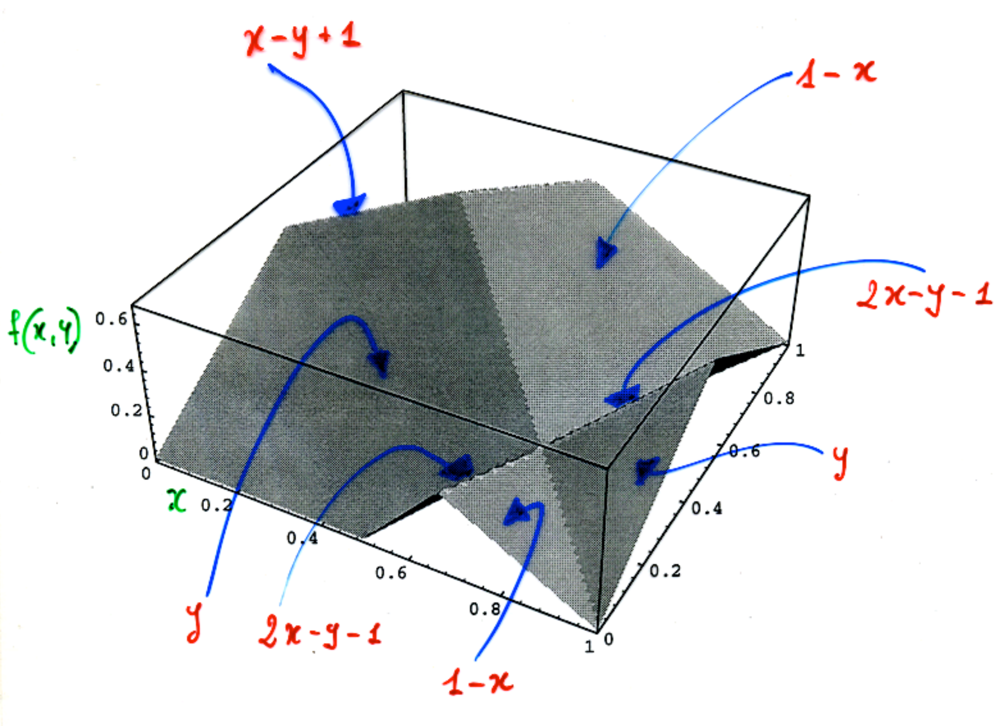


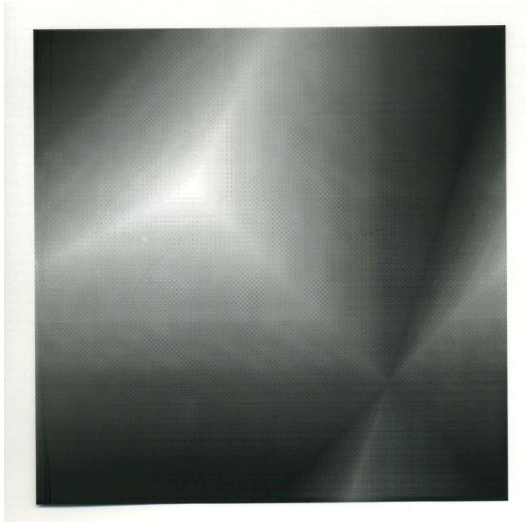




The free two-generator MV-algebra F_2

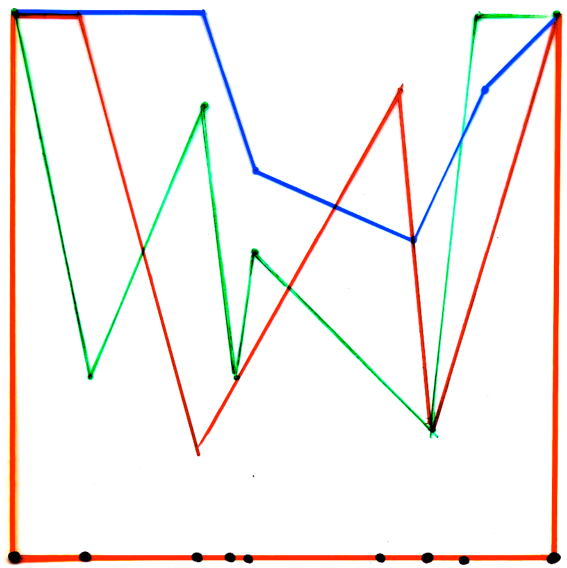
F_2 consists of all piecewise linear continuous $[0,1]$*-valued functions with integer coefficients, defined over the* $[0,1]^2$.
$F_1 \subseteq F_2$ *by cylindrification*.

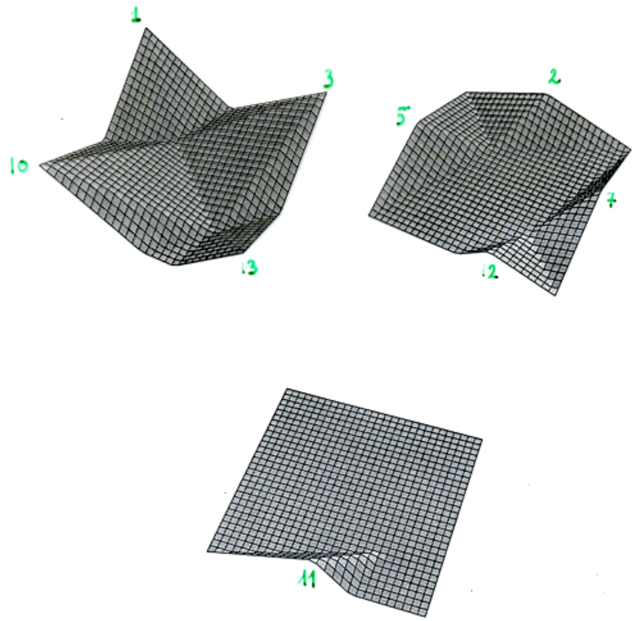




The free MV-algebra F_ω over countably many generators

F_ω is the union of all free algebras F_n. A function $f : [0,1]^\omega \rightarrow [0,1]$ is in F_ω iff it is piecewise linear with integer coefficients and has a finite number of variables. $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \qquad F_\omega = \bigcup_i F_i$





• • •

the free AF-algebra M

THEOREM (D.M., J.Functional Analysis 1986) *The AF-algebra* M *corresponding to* F_ω *via* Γ *and* K_0 *has the following properties:*

Quotients of M *precisely coincide with AFl-algebras.*

Every AF algebra is a subalgebra of a quotient of M.

Every AF algebra with comparability of projections is a quotient of M *by a primitive ideal.*

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this was the first transfer of mathematical information

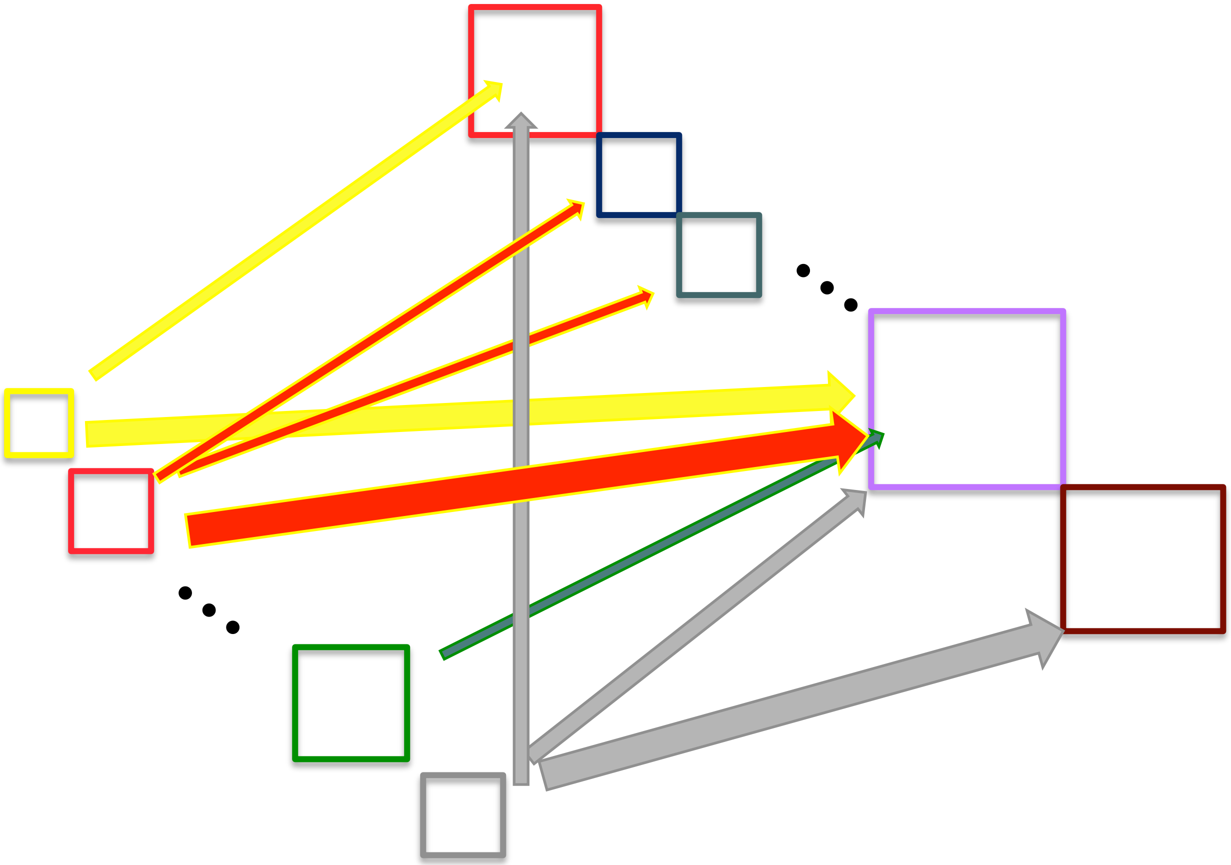
from

MV-algebras and Łukasiewicz logic

to

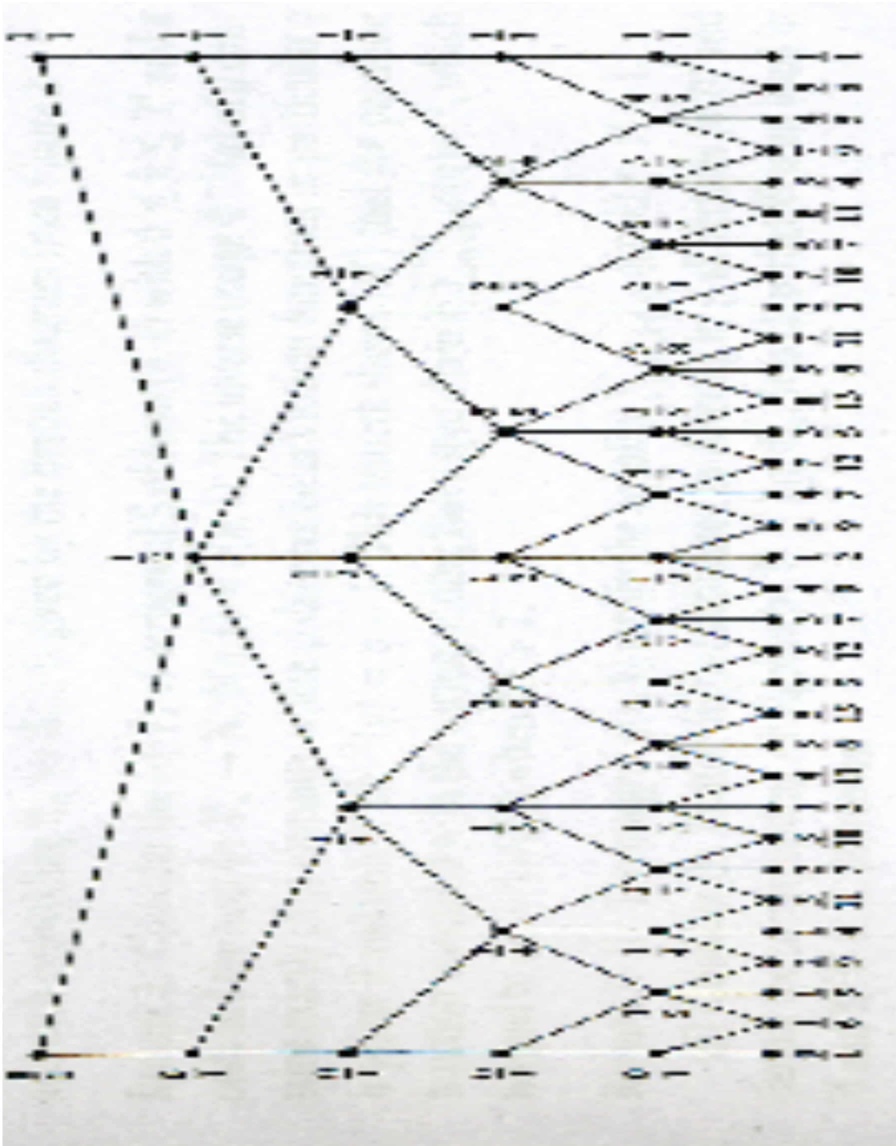
functional analysis

a morphism between finite-dimensional C*-algebras



the thicker the arrow, the higher the multiplicity

F. Boca’s algebra, Can. J. of Math., 60.5, 2008, p.977



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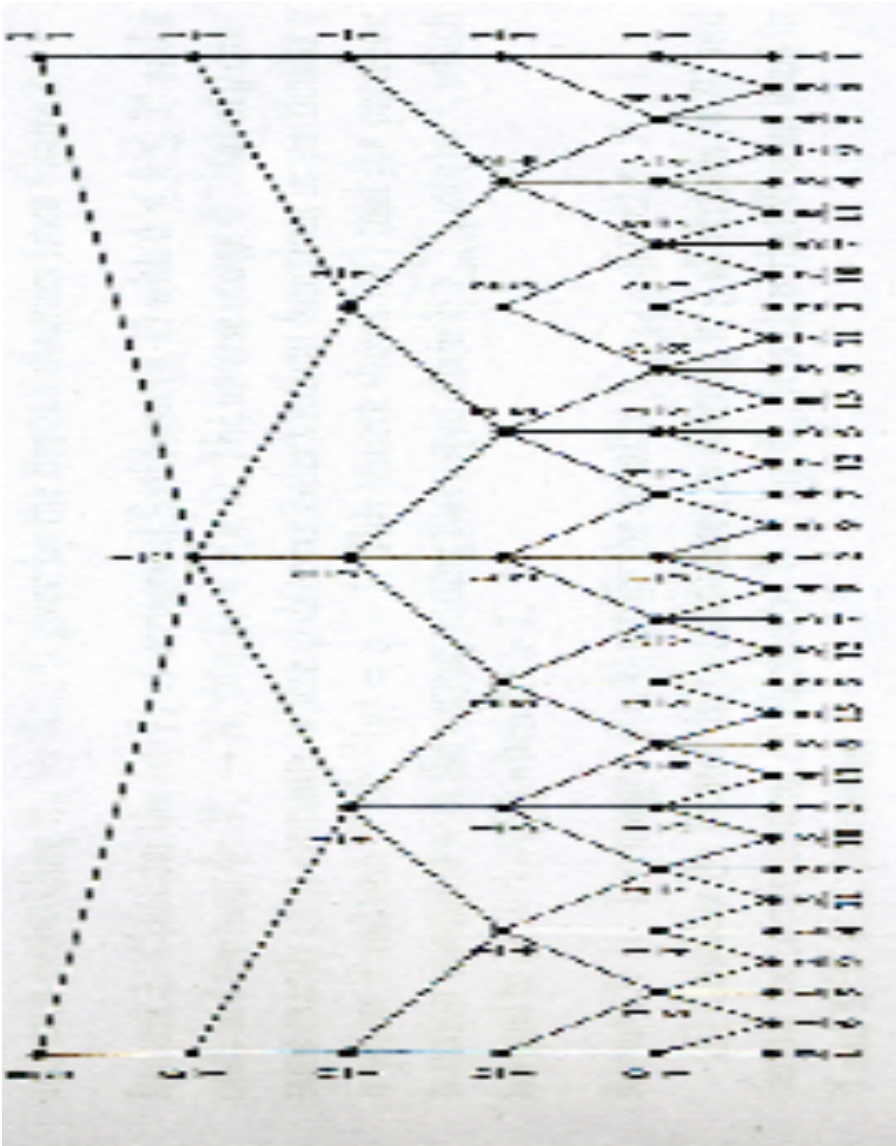
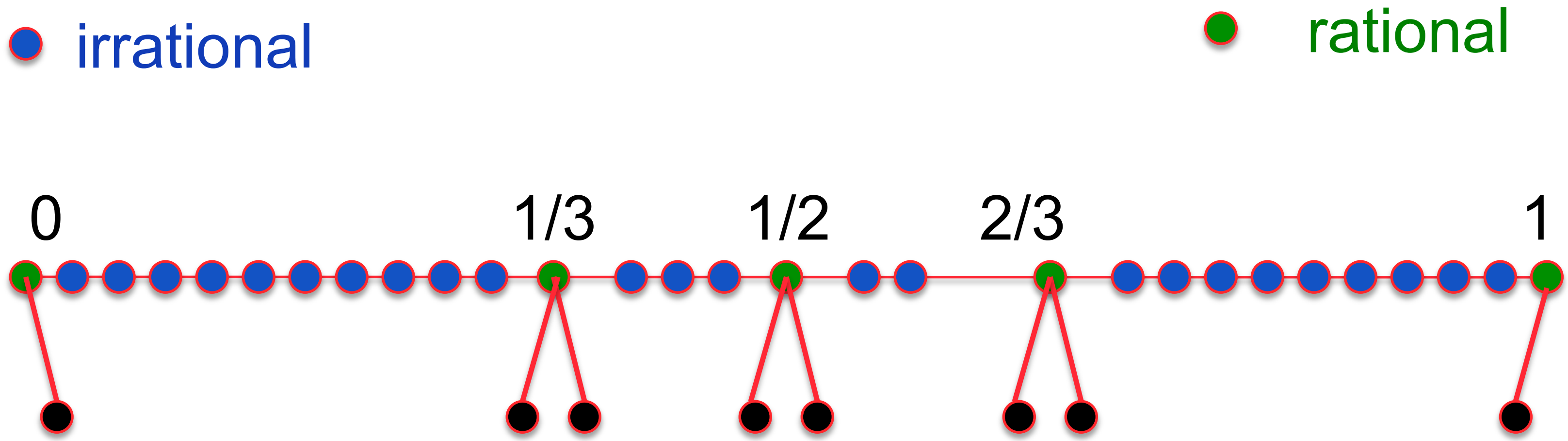


THEOREM (D.M. Rend. Accad. Lincei, 20, 2009)

F. Boca’s diagram defines precisely the AFl-algebra M_1 *corresponding to the free MV-algebra* F_1

M_1 had been introduced in 1988 (D.M., Advances in Mathematics, vol. 68)

the primitive ideal space of M_1
= the prime ideal space of F_1



each point at level 0 stands for a maximal ideal

below each irrational point there is no primitive ideal

below each rational point ≠ 0,1 there are two primitives

below 0 and 1 there is precisely one primitive ideal

other basic properties of M_1

THEOREM (C.Eckardt, Math. Scand. 108, 2011) *One can construct a noncommutative Gauss map on* M_1.

THEOREM (D.M., Milan J.Math.79, 2011) *Up to isomorphism, the infinite-dimensional simple quotients of* M_1 *coincide with the Effros-Shen AF-algebras* F_θ.

Each primitive ideal J of M_1 *is* essential *(every nonzero ideal of* M_1 *has a nonzero intersection with J)*

Second Application: AF-algebraic computation from the logic-deductive machinery of Łukasiewicz logic

The freeness properties of the MV-algebras L(M) and $L(M_1)$ yield a natural **word problem for every AF*l*-algebra B**, because B is automatically a quotient of M.

Given two terms in the language of MV-algebras, **the problem asks to decide whether they code the same Murray-von Neumann equivalence class of projections of B**. This mimics the classical definition of the word problem of a group presented by generators and relations.

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Łukasiewicz chain $\{0,1/n,...,1\}$	$M_n(\mathbf{C})$, the $n \times n$ complex matrices
free with $\aleph_o$ generators,	the universal AF C*-algebra M
free with one generator	the Farey algebra M_1

Polytime Decidability and Gödel Incompleteness

(D.M., Advances in Mathematics 2018, to appear)

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THEOREM (D.M., J.Functional Analysis, 1986) *If an AFl-algebra B is Gödel incomplete, then it must have a nontrivial ideal.*

Third Application: Stable Consequence and Differential Semantics

A deep theorem ...

THEOREM (ROSE-ROSSER 1958) **Syntactic** *tautologies (i.e., those formulas obtainable from the axioms via substitution and Modus Ponents) coincide with* **semantic** *tautologies (i.e., those formulas which are evaluated to* 1 *by every valuation)*

COROLLARY (D.M., Theoretical Computer Science, 1987) *The tautology problem of Łukasiewicz logic is coNP-complete*.

... and an embarassing fact

DEFINITION *A formula X is a* syntactic consequence *of a set K of formulas if it is obtainable from the axioms and from a finite subset K' of K via modus ponens*.

DEFINITION *A formula X is a (Bolzano-Tarski)* semantic consequence *of a set K of formulas every valuation giving value* 1 *to all formulas in K also gives value* 1 *to X*.

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FACT *Syntactic consequences do not coincide with Bolzano-Tarski semantic consequences…*

A deep theorem and an embarassing fact

The sentence X follows logically *from the sentences of the class K if and only if every model of the class K is also a model of the sentence X.*†

† After the original of this paper had appeared in print, H. Scholz in his article 'Die Wissenschaftslehre Bolzanos, Eine Jahrhundert-Betrachtung', *Abhandlungen der Fries'schen Schule*, new series, vol. 6, pp. 399–472 (see in particular p. 472, footnote 58) pointed out a far-reaching analogy between this definition of consequence and the one suggested by B. Bolzano about a hundred years earlier.

A.Tarski, Logic, Semantics, Metamathematics, p.417

A main service of Łukasiewicz logic to many-valued reasoning is given by its **differential** consequence relation, that naturally goes beyond the classical Bolzano-Tarski paradigm

in classical mechanics: position alone does not control the **dynamic evolution** of a material point

full information is given by **position**

together with **speed**

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in Łukasiewicz logic point-evaluation alone does not control the **deductive closure** of a set of premises

analysis of traffic jam

this picture does not tell us
which cars are standing, and
which cars are in motion;
the shutter was open
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analysis of traffic jam

this picture does not tell us which cars are standing, and which cars are in motion; the shutter was open virtually only for an instant t_0

this picture does: the shutter was open for a (short) time interval $[t_0,t_1]$

logical consequence

in classical logic one decides whether f follows from g by **pointwise** evaluation of f and g:
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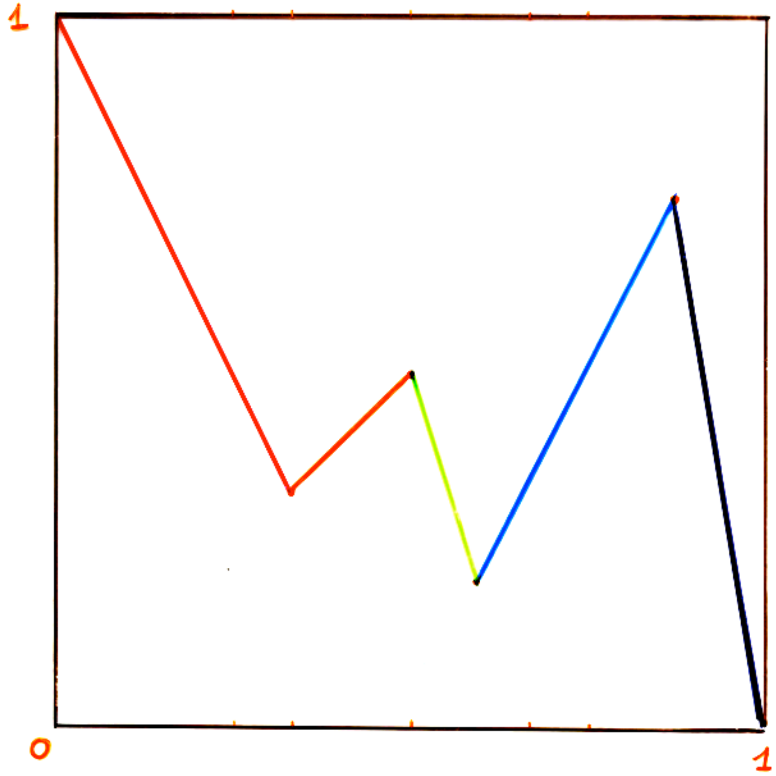
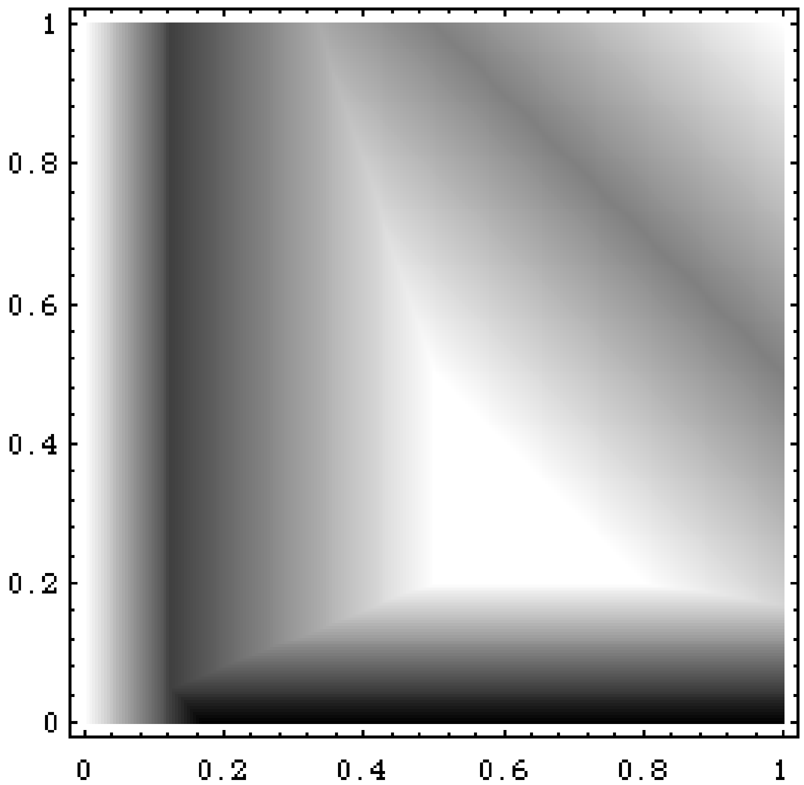
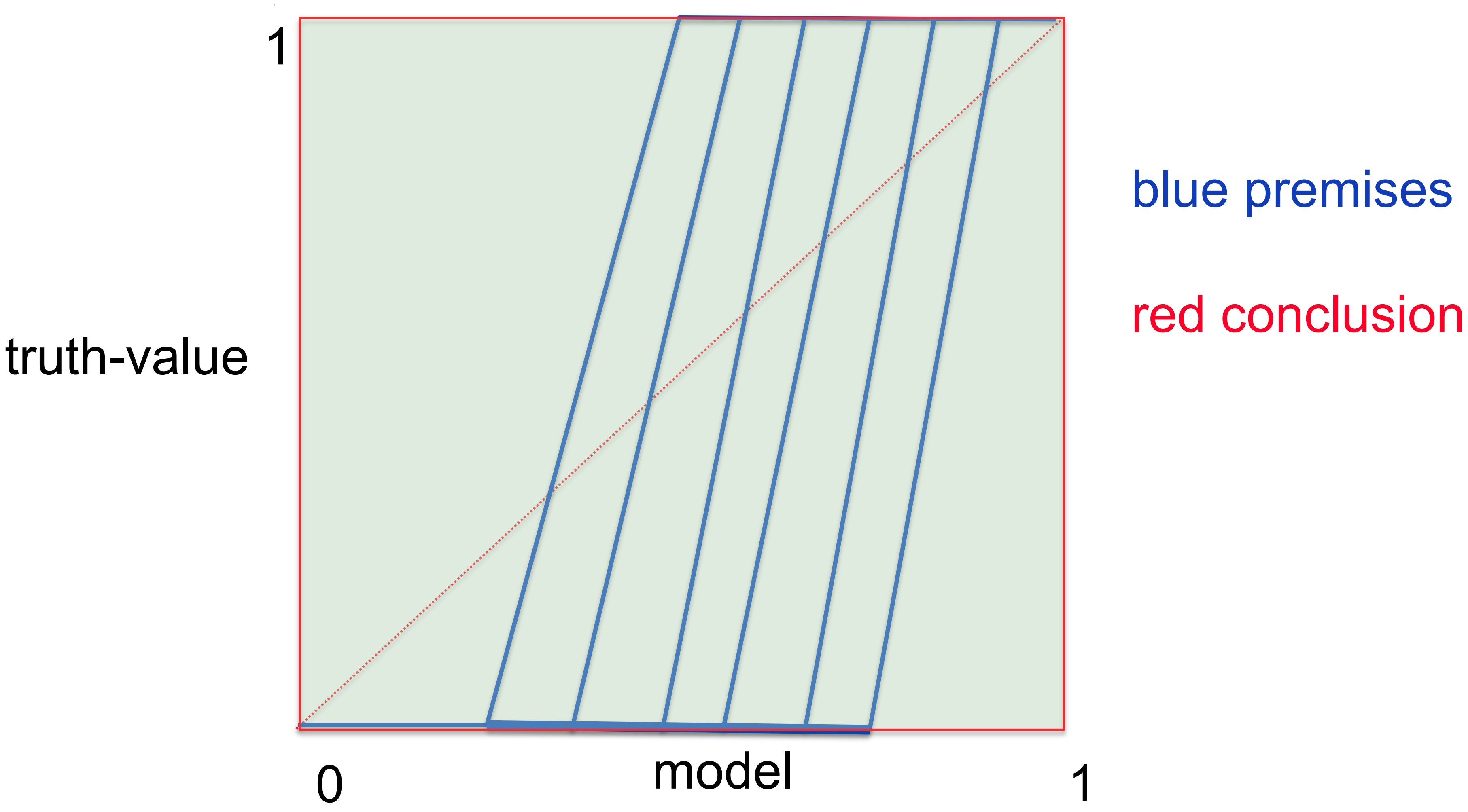
to achieve completeness in [0,1]-valued logic, one needs **differential evaluations**

Recall: formulas code McNaughton functions. Each McNaughton function has all directional derivatives





failure of completeness



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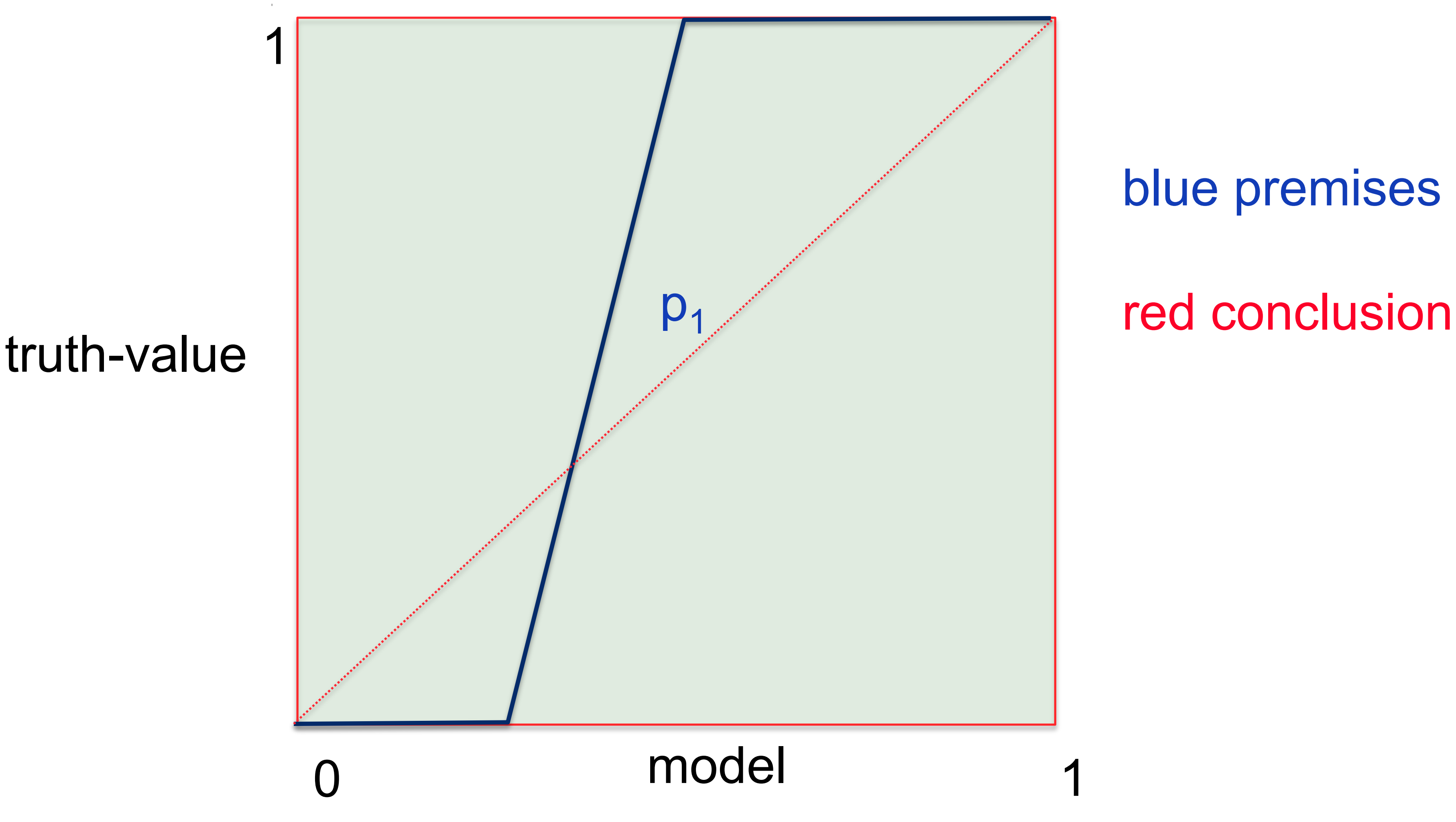
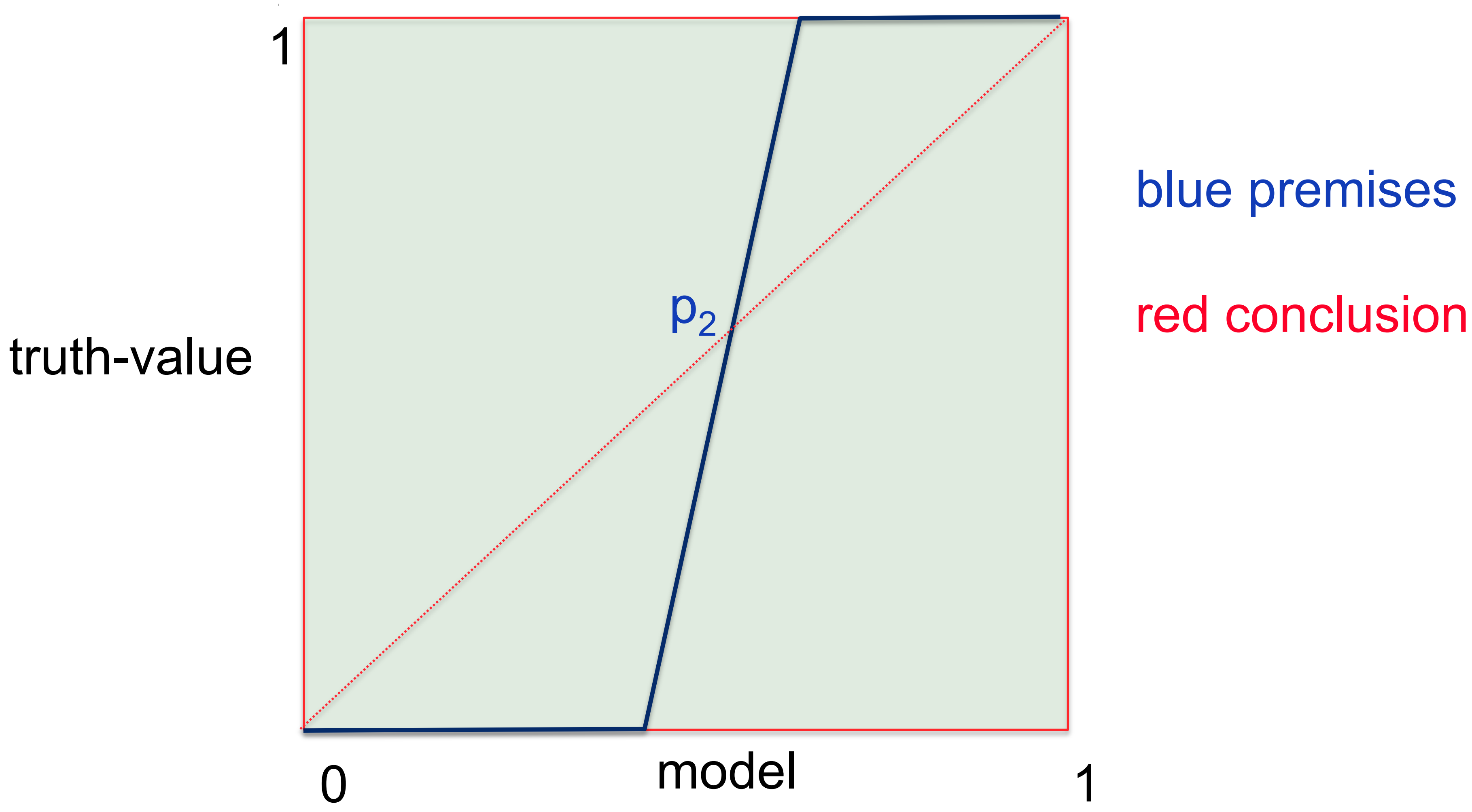


failure of completeness



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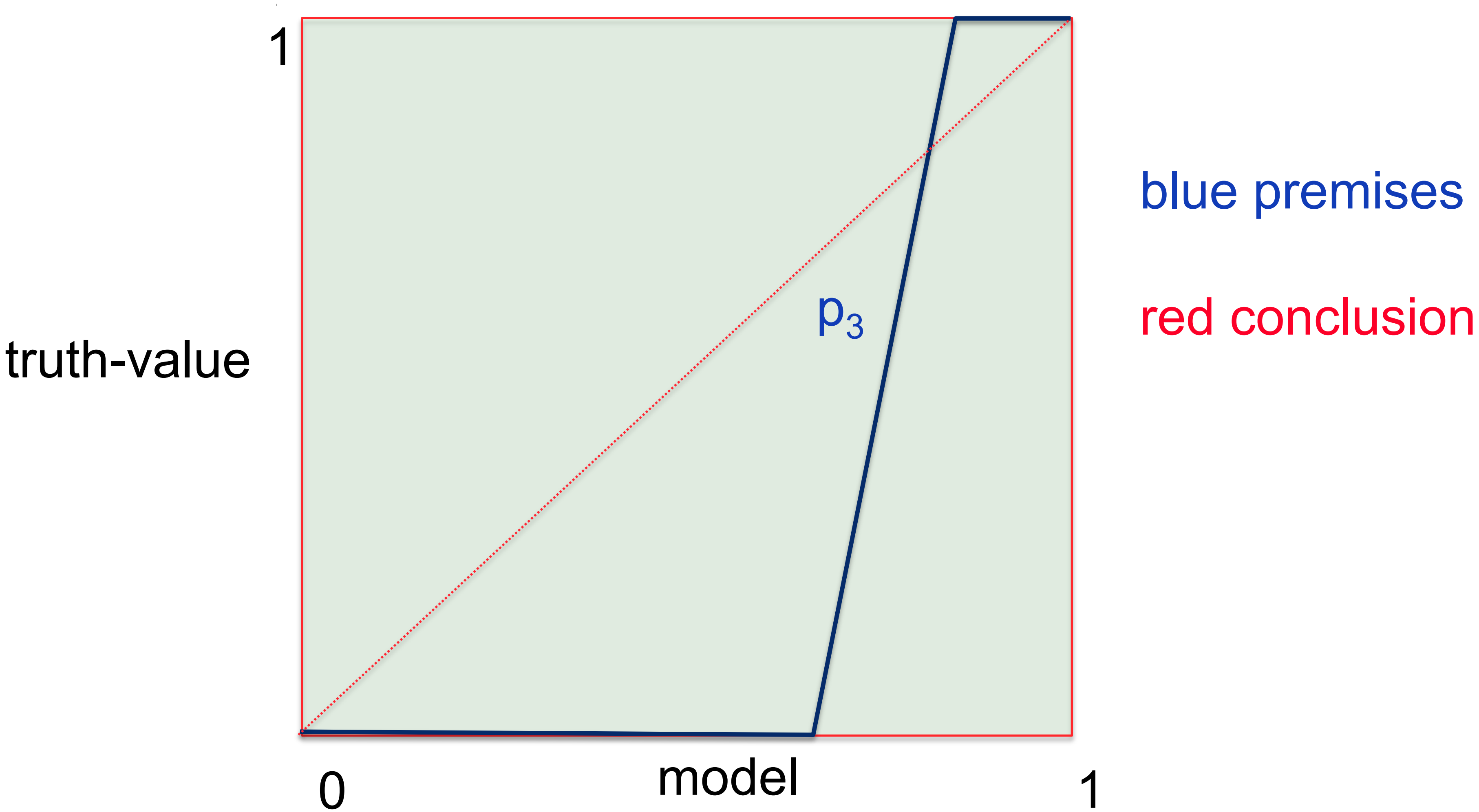

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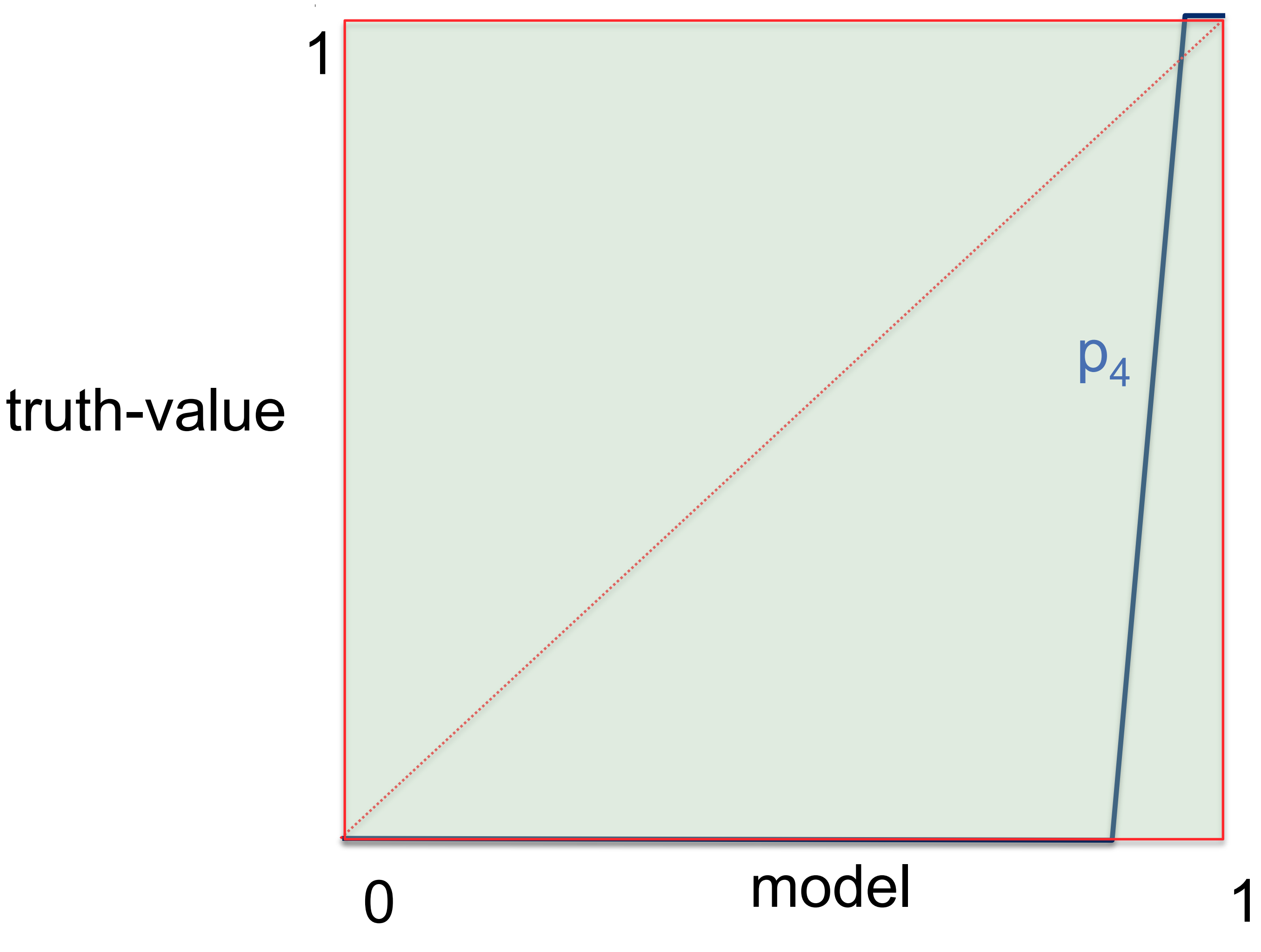


blue premises

red conclusion

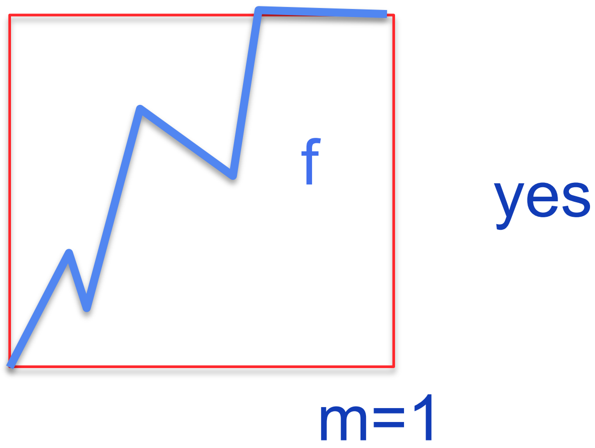
stable consequence goes beyond the Bolzano-Tarski paradigm

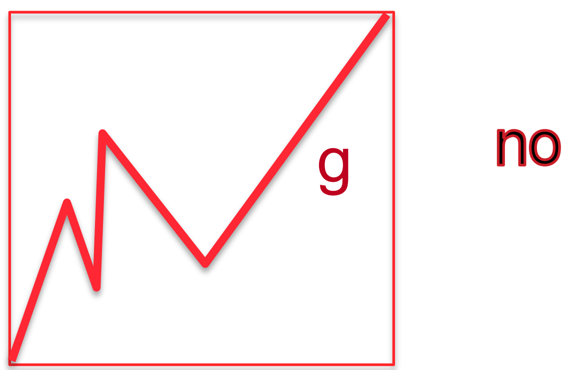
let P be the set of formulas which are true for all models in a left neighbourhood of m=1

f is a **stable** consequence of P iff f is true for all models in a left neighbourhood of m=1

g is a **Bolzano-Tarski** consequence iff g(1)=1

The red function g is a Bolzano-Tarski consequence of P without being a stable consequence of P





classical consequence (static evaluation)

f is a ***Bolzano-Tarski consequence*** *of a set P of premises if for every model (= valuation) m,*

$$p(m) = 1 \quad \textit{for all } p \textit{ in } P,$$

$$\Downarrow$$

$$f(m)=1$$

stable consequence (differential evaluation)

*f is a **stable consequence** of a set P of premises if for every valuation m and direction d in the valuation space,*

$$p(m) = 1 \quad \textit{and} \quad \partial p(m)/\partial d = 0 \quad \textit{for all} \quad p \quad \textit{in } P$$

$$\Downarrow$$

$$f(m) = 1 \quad \textit{and} \quad \partial f(m)/\partial d = 0$$

stable consequence (differential evaluation)

this differential condition makes sense because the valuation space is a cube $[0,1]^n$*, and formulas code* $[0,1]$*-valued functions over* $[0,1]^n$ *that have all directional derivatives (McNaughton theorem, 1951).*

stable consequence (differential evaluation)

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Stable consequence prescribes that whevever all premises are stably true under an infinitesimal perturbation of their evaluation, then so is the conclusion.

stable consequence (differential evaluation)

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Stable consequence prescribes that whevever all premises are stably true under an infinitesimal perturbation of their evaluation, then so is the conclusion.

In the boolean fragment of Łukasiewicz logic, stable consequence trivially boils down to Bolzano-Tarski consequence

the completeness theorem

THEOREM (D.M. 2013, Trends in Logic, honoring Petr Hàjek)

f is a stable consequence of a set P of premises iff it is a syntactic consequence of P iff Modus Ponens derives f from a finite subset of P and the tautologies

COROLLARY *Stable consequence is* ***finitary****: if f is a stable consequence of P then f is a stable consequence of a finite subset of P. The converse is trivial.*

in Łukasiewicz logic, Bolzano-Tarski semantic consequence differs from syntactic consequence: the latter captures all dynamic aspects of consequence; the former does not

stable consequence is the natural notion of semantic consequence, capturing all dynamical features of consequence in Łukasiewicz logic, and agreeing with syntactic consequence

Concluding remarks: towards the second century of Łukasiewicz logic

Łukasiewicz logic and MV-algebras in mathematics

- Dobrushin-Berlekamp coding with feedback
- Ulam-Rényi game on binary search wth errors
- lattice-ordered abelian groups
- AF C*-algebras
- polyhedral geometry
- toric varieties and desingularization
- de Finetti probability theory
- geometry of the affine group over the integers
- inverse semigroups and coordinatization
- Riesz spaces
- semirings, tropical mathematics

an incomplete list of recent contributors

MV and Categories, Morita equivalence, duality, sheafs, etc.: Caramello, Gehrke, Lawson, Marra, Russo, Scott, Spada
MV and Probability: Flaminio, Gerla, Keimel, Montagna, Riečan
MV and Games: Kroupa, Teheux
MV and Discrete Dynamical Systems: Cabrer
MV and Riesz spaces: Cabrer, Di Nola, Lapenta, Leustean
MV and Multisets: Cignoli, Marra, Nganou
MV and Proof-theory: Cabrer, Ciabattoni, Jeřábek, Metcalfe
MV and Modal logic, Belief: Flaminio, Godo, Kroupa, Teheux
MV and Semirings, tropical mathematics: Belluce, Di Nola, Kala
MV and Quantum structures: Dvurečenskij, Pulmannová

three selected duality theorems

THEOREM (R.Cignoli, E.Dubuc, D.M., J. Pure and Applied Algebra, 2004) *Locally finite MV-algebras are dually equivalent to multisets*.

THEOREM (L. Spada, V.Marra, Studia Logica, 2012) *Finitely presented MV-algebras are dually equivalent to rational polyhedra and piecewise linear cotinuous maps with integer coefficients*.

THEOREM (L. Reggio,V.Marra, Advances in Mathematics, 2017) *MV-algebras equipped with just one additional infinitary operation form an equational class which is dually equivalent to the category of compact Hausdorff spaces with continuous maps*.

THANK YOU

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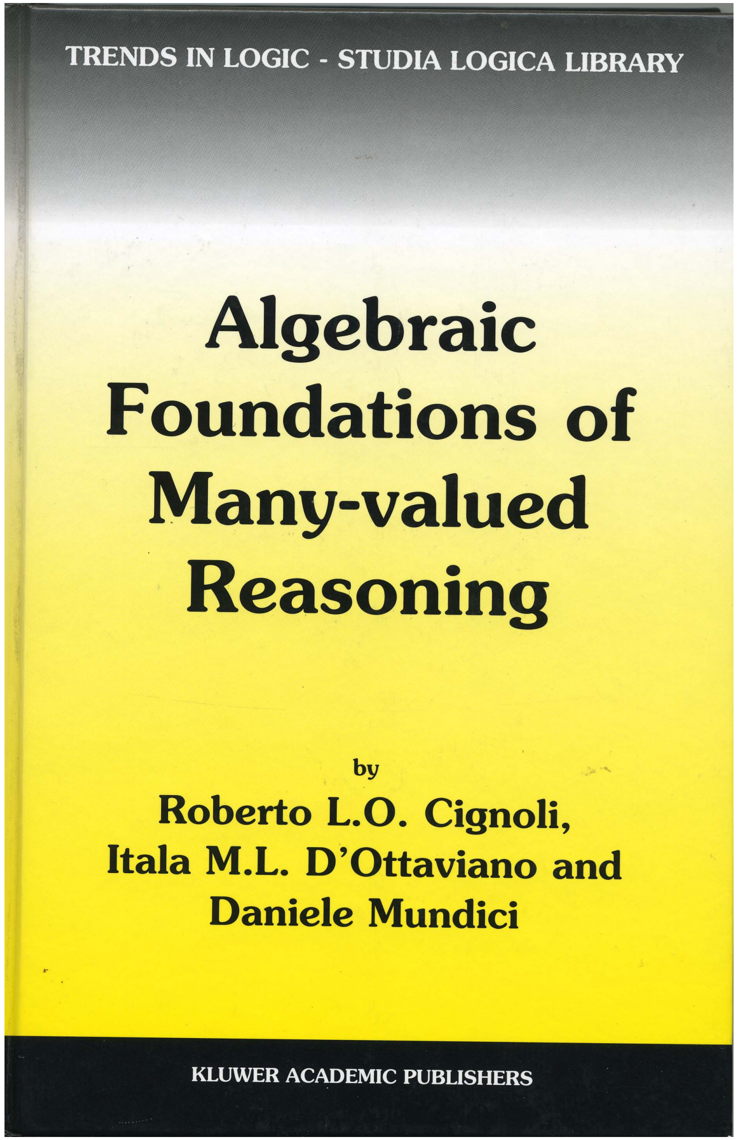

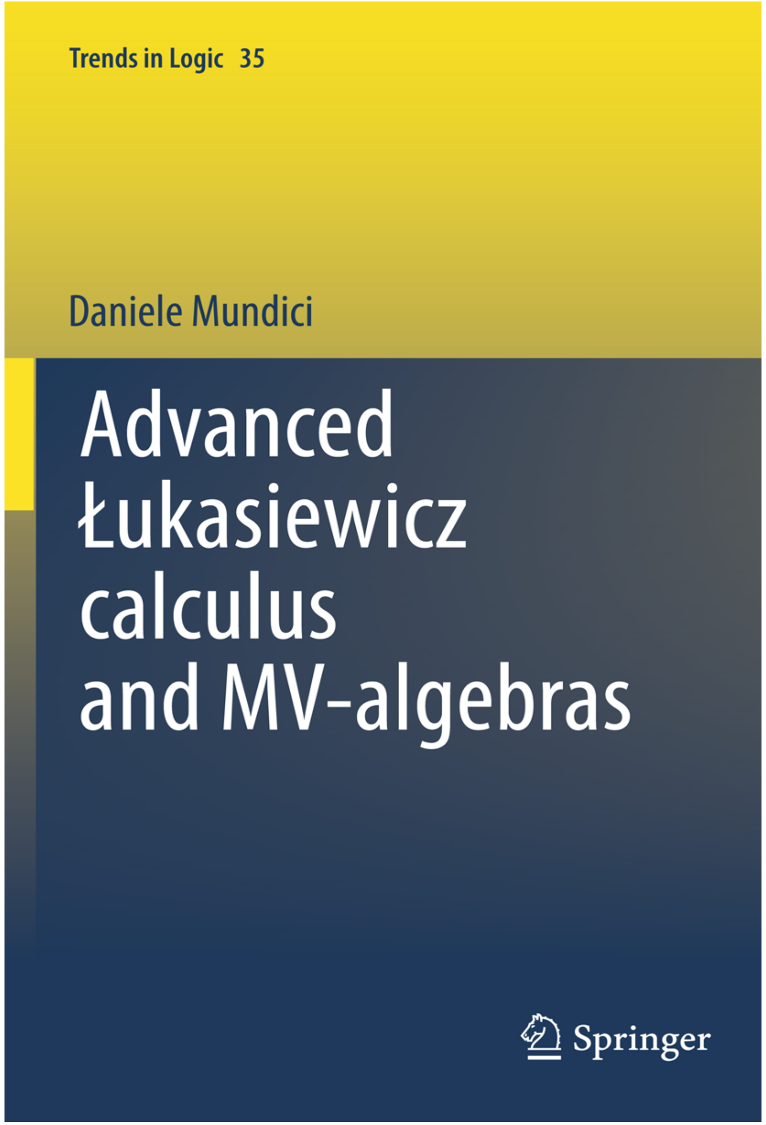

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