Model theoretic ampleness

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Plan of tutorial

Explain 'ampleness' and its connection to projective spaces

1. Background and definitions

   Strongly minimal structures, dimension, modularity, independence

2. Stability and ampleness (start over.....)

   Free groups, simplicial complexes, projective spaces

3. Recent examples of ample structures

   Constructions of ample structures by Hrushovski method

What's the point of model theory

Model theory is looking for results of the form:

If a mathematical structure has a certain formal model theoretic property, then......

....it is vector space 'in disguise', e.g. all groups in this setting are abelian.

or:

...it is field 'in disguise', e.g. all one-dimensional sets 'are' algebraic curves.

Zilber's Conjecture A strongly minimal structure is either trivial, vector space-like or field-like.

Aim: Explain this....

Basics

Model theory considers structures in a fixed 1^{st}-order language L, e.g.

- $L_{gph} = \{E\}$ for graphs
- $L_{gp} = \{\cdot, 1\}$ for groups
- $L_{rng} = \{\cdot, +, 0, 1\}$ for rings and fields
- $L_{F-vsp} = \{+, 0, \lambda_a \colon a \in F\}$ for F-modules or vector spaces.

If M is an L-structure, then all symbols of L have fixed interpretation in M. So for an L-sentence φ it makes sense to consider

$$M \models \varphi$$

Definable sets

We often allow ourselves to name elements from M as parameters. So may consider $L(M)$-sentence $\varphi(\bar{a})$ for $\bar{a} \subset M$ and

$$M \models \varphi(\bar{a})$$

For an $L(M)$-formula $\varphi(x_1, \ldots x_n)$ consider its set of realizations

$$X_\varphi := \varphi(M) = \{(a_1, \ldots, a_n) \in M^n \colon M \models \varphi(\bar{a})\} \subseteq M^n.$$

Call X_φ a *definable subset* of M (or M^n).

Definition

A definable set in an L-structure M is a set of the form $\varphi(M)$ for some $L(M)$-formula φ.

Definable sets

Language of groups L_{gp} $\varphi(x_1, x_2)$: $x_1 \cdot x_2 = x_2 \cdot x_1$.

If G is an L_{gp}-structure, $a \in G$, then $\varphi(x_1, a)$: $x_1 \cdot a = a \cdot x_1$.

If G is a group, $\varphi(x_1, a)$ defines the *centralizer* of a in G.

Language of graphs L_{gph} $\varphi(x_1, x_2)$: $\exists x_3.E(x_1, x_3) \wedge E(x_3, x_2)$.

If Γ is an L_{gph}-structure, $a \in \Gamma$, then $\varphi(x_1, a)$: $\exists x_3.E(x_1, x_3) \wedge E(x_3, x_a)$.

If Γ is a graph, $\varphi_{gph}(x_1, a)$ defines the set of elements of distance at most 2 from a.

Note that a connected component of a graph is a definable set if and only if the component has *bounded diameter*.

Structures in a structure

Suppose M is an L-structure.

We say that $(G, \cdot)$ is a *definable group* in M if the underlying set $G \subset M^n$ and the graph of the multiplication function (as a subset of M^{3n}) and of inversion are definable sets in M.

Similarly for fields, graphs, etc.

How can we determine what subsets or structures are definable?

Definable sets

Definable subsets can be very complicated

The integers $(\mathbb{Z}, +, \cdot)$ as an L_{rng} structure:

for any polynomial $P(x_1, \ldots, X_n; Y_1, \ldots Y_m) \in \mathbb{Z}[\bar{X}; \bar{Y}]$ there is a definable subset $X_p \subset \mathbb{Z}^n$ given by

$$\varphi_P(\bar{x}) : \exists y_1, \ldots y_m . P(\bar{x}, \bar{y}) = 0$$

Is $X_P = \emptyset$?

$\longrightarrow$ arithmetical hierarchy.................

Quantifier elimination

In some cases, there is a convenient description of the definable sets:

Example (Chevalley-Tarski)

Let F be an algebraically closed field considered as an L_{rng}-structure. Then every definable set is a boolean combination of zero-sets of polynomials.

In other words, every definable subset can be defined using a quantifier-free formula.

Definition

An L-structure M admits *quantifier elimination* if every definable set can be defined using a quantifier free formula.

Quantifier elimination

Example (Vector spaces)

Consider an infinite F-vector space V as L_{F-vsp}-structure.

Quantifier free formulas express that an element is (or is not) a specific linear combination of other elements.

Therefore, any existential formula defines a set that can be defined without quantifiers. Hence get quantifier elimination for V in L_{F-vsp}.

Graphs

Example (Random Graph)

Let Γ be the countable random graph considered as L_{gph}-structure,

i.e. Γ is characterized by the following axioms:
For all finite disjoint subsets $A, B \subset \Gamma$ there is some vertex $c \in \Gamma$ with edges to all $a \in A$ and no edges to any $b \in B$.

This determines Γ up to isomorphism.

In Γ the formula $\varphi_{gph}(x_1, x_2) : \exists x_3 E(x_1, x_3) \wedge E(x_3, x_2)$ is true for all x_1, x_2, so equivalent to the formula $x_1 = x_1 \vee x_2 = x_2$.
Similarly for $\neg\varphi_{gph}(x_1, x_2)$.

Thus, Γ admits quantifier elimination.

Example (Infinite trees)
Let Γ_{tr} be an infinite tree with infinite valencies. Then the subset of Γ_{tr}^2 defined by $\varphi_{gph}(x_1, x_2) : \exists x_3 E(x_1, x_3) \wedge E(x_3, x_2)$ cannot be defined by a quantifier free formula.

However, if we extend the language by binary predicates $d_k, k \in \mathbb{N}$, denoting the graph distance, then again this set is quantifier free definable in this extended language.

In this extended language, Γ_{tr} has quantifier elimination.

Strongly minimal sets and structures

Since polynomials in one variable have only finitely many zeros, it follows from quantifier elimination that any definable subset $X \subset F$ for an algebraically closed field F is finite or cofinite.

Definition

An infinite structure M is called *strongly minimal* if every definable subset of M is finite or cofinite in M (and the same is true in any elementary extension of M).

An infinite definable subset $X \subset M^n$ is called *strongly minimal* if every definable subset of X is finite or cofinite in X.

Strongly minimal sets

Examples

1. For Γ_{tr} the infinite tree, a vertex in Γ_{tr}, the definable set $E(x, a)$ of neighbours of a is strongly minimal.

2. If V is an F-vector space, then V has quantifier elimination in the language L_{F-vsp}, and hence is strongly minimal.

3. Algebraically closed fields

Zilber conjectured that these are *essentially* all examples.

Two complementary aspects of model theory

I. Given an L-structure M, determine the definable subsets of M

Investigate the properties of the definable sets, and hence the model theoretic properties of M.

Conversely:

II. Given some model theoretic properties, classify the structures having these properties

Motivating question:

Is there a model theoretic property that characterizes projective spaces?

Geometry on strongly minimal sets Zilber's Conjecture

Strongly minimal sets come with a notion of geometry and dimension:

Definition

Let M be an L-structure, $A \subset M$. We say that $\bar{b} \in M^n$ is *algebraic over* A, if there is an $L(A)$-formula $\varphi(x)$ realized by $\bar{b}$ such that $\varphi(M)$ is finite. Write $\bar{b} \in \mathrm{acl}(A)$. We call A *algebraically closed* if $\mathrm{acl}(A) = A$.

Examples

1. Let V be an infinite F-vector space, $a \in V, A \subset V$. By quantifier elimination,

$$a \in \mathrm{acl}(A) \text{ if and only if } a \in \langle A \rangle_F.$$

2. Let K be an algebraically closed field, $a \in K, A \subset K$. Then

$$a \in \mathrm{acl}(A) \text{ if and only if } a \in \mathrm{acl}^K(A).$$

Dimension

If M is *strongly minimal*, then the *exchange property* holds:

$$\text{If } a, b \notin \operatorname{acl}(A) \text{ and } a \in \operatorname{acl}(Ab), \text{ then } b \in \operatorname{acl}(Aa).$$

This leads to a well-defined notion of independence and dimension:

If M is strongly minimal, then $a \in M$ is independent from $A \subset M$ if $a \notin \operatorname{acl}(A)$.

Say that $a, b \in M \setminus \operatorname{acl}(A)$ are independent over $A \subset M$ if $a \notin \operatorname{acl}(Ab)$.

Thanks to the exchange property this notion is symmetric. Write

$$a \underset{A}{\perp} b.$$

Independence and dimension

Note that for F-vector spaces, a is independent from A (in the sense of model theory) if and only if a is linearly independent from A, i.e. $a \notin \langle A \rangle_F$. Similarly for algebraically closed fields.

If M is a strongly minimal L-structure, for $A \subset M$ we put

$$\dim(A) = \min\{|A_0| \colon A_0 \subset A, A \subset \operatorname{acl}(A_0)\}.$$

For F-vector spaces and algebraically closed fields, the model theoretic dimension agrees with the (linear) algebraic dimension.

Triviality

A strongly minimal structure M is called *trivial* if for all $a, b, c \in M, A \subset M$

$$c \in \operatorname{acl}(abA) \quad \text{implies} \quad c \in \operatorname{acl}(aA) \cup \operatorname{acl}(bA).$$

Non-example

If V is a vector space, $a, b \in V$ are linearly independent and $c = a + b$, then

$$c \in \operatorname{acl}(ab) \setminus (\operatorname{acl}(a) \cup \operatorname{acl}(b)).$$

Examples

- Any set in the empty language $L_=$ is trivial (and strongly minimal).
- If $a \in \Gamma_{tr}$, the set of neighbours of a is a trivial strongly minimal set.

Modularity

A strongly minimal structure M is called *modular* if for all $A, B \subset M$ with $A = \mathrm{acl}(A), B = \mathrm{acl}(B)$ the dimension satisfies

$$\dim(A) + \dim(B) = \dim(A \cup B) + \dim(A \cap B)$$

and *locally modular* if the above holds whenever $\dim(A \cap B) > 0$.

Example

Clearly, vector spaces are modular.

(Non-)modularity

$$\dim(A) + \dim(B) = \dim(A \cup B) + \dim(A \cap B)$$

Algebraically closed fields are not modular

Let $a, b, c \in \mathbb{C}$ be independent elements. Consider $p = (a, b)$ and $c \cdot p = (ca, cb)$ in $\mathbb{C}^2$.

Then $\dim(p) = 2 = \dim(c \cdot p)$ and $\dim(p, c \cdot p) = 3$. So

$$\dim(p) + \dim(c \cdot p) = 2 + 2 = 4 > \dim(p, c \cdot p) = 3.$$

Not even locally modular

Let $d \in \mathbb{C}$ be independent from a, b, c. Then

$$\dim(p, d) + \dim(c \cdot p, d) = 3 + 3 = 6 > \dim(p, c \cdot p, d) + \dim(d) = 4 + 1 = 5.$$

Zilber's Conjecture

With this terminology Zilber's Conjecture can be stated as follows:

Conjecture

If the geometry of a strongly minimal structure M is not locally modular, then there is a definable field in M.

This conjecture was disproved by Hrushovski:

Theorem (Hrushovski)

There are strongly minimal structures which are not locally modular and do not define any group or field.

We'll get back to these counterexamples later.

Independence and dimension

We can extend the notion of independence to subsets of a strongly minimal structure in the following way:

$$A \underset{B}{\perp} C$$

if and only if

$$\dim(A/B) = \dim(A/BC)$$

where

$$\dim(A/B) = \dim \operatorname{acl}(AB) - \dim \operatorname{acl}(B).$$

Modularity and independence

So M is modular

if and only if for all algebraically closed A, B

$$\dim(A) + \dim(B) = \dim(A \cup B) + \dim(A \cap B)$$

if and only if for all algebraically closed A, B

$$A \underset{A \cap B}{\downarrow} B.$$

Ampleness

Finally, here is our protagonist:

Definition (Pillay)

A strongly minimal structure M is k-ample for some $k \geq 1$ if there are tuples $\bar{a}_0, \ldots, \bar{a}_k \subset M$ such that (possibly after naming parameters) for all $0 \leq i < k$ the following hold:

- $\bar{a}_0 \ldots \bar{a}_i \not\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}} \bar{a}_{i+1} \ldots \bar{a}_k$;
- $\bar{a}_0 \ldots \bar{a}_{i-1} \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}_{\bar{a}_i} \bar{a}_{i+1} \ldots \bar{a}_k$;
- $\mathrm{acl}(\bar{a}_0 \ldots \bar{a}_{i-1}\bar{a}_i) \cap \mathrm{acl}(\bar{a}_0 \ldots \bar{a}_{i-1}\bar{a}_{i+1}) = \mathrm{acl}(\bar{a}_0 \ldots \bar{a}_{i-1})$.

1-ample = non-modular

Definition

A strongly minimal structure M is 1-ample if there are tuples $a_0, a_1 \in M$ such that (possibly after naming parameters)

- $a_0 \not\perp a_1$;
- $\mathrm{acl}(a_0) \cap \mathrm{acl}(a_1) = \mathrm{acl}(\emptyset)$.

Thus a 1-ample structure M is *not modular*: $\mathrm{acl}(a_0), \mathrm{acl}(a_1)$ witness the non-modularity of M.

Conversely, if $A \not\perp_{A \cap B} B$, then after naming $A \cap B$ the sets A, B witness 1-ampleness.

Ampleness and algebraically closed fields

Pillay showed that algebraically closed fields are k-ample for all k.

Question

Does a strongly minimal structure which is k-ample for all k define an infinite field?

Outlook

- Give metric interpretation of independence
- Explain the intuition behind the notion of ampleness
- Explain ampleness in non-abelian free groups
- Sketch construction of ample strongly minimal structures