Model theoretic ampleness

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Udine, July 2018

Plan of tutorial

Explain 'ampleness' and its connection to projective spaces

1. Background and definitions

   Strongly minimal structures, dimension, modularity, independence

2. Stability and ampleness

   Free groups, simplicial complexes, projective spaces

Today:

3. Recent examples of ample structures without fields

   Constructions of ample structures by Hrushovski method

Recall

Definition (Pillay-Evans)

A *stable* structure M is k-ample for some $k \geq 1$ if there are tuples $\bar{a}_0, \dots, \bar{a}_k \subset M$ such that (possibly after naming parameters) for all $0 \leq i < k$ the following hold:

- $\bar{a}_0 \dots \bar{a}_i \not\mathop{\smile\kern-0.65em\mid}\, \bar{a}_{i+1} \dots \bar{a}_k$;
- $\bar{a}_0 \dots \bar{a}_{i-1} \mathop{\smile\kern-0.65em\mid}_{\bar{a}_i} \bar{a}_{i+1} \dots \bar{a}_k$;
- $\mathrm{acl}(\bar{a}_0 \dots \bar{a}_{i-1}\bar{a}_i) \cap \mathrm{acl}(\bar{a}_0 \dots \bar{a}_{i-1}\bar{a}_{i+1}) = \mathrm{acl}(\bar{a}_0 \dots \bar{a}_{i-1})$.

Independence

For finite sets A, B, C contained in some metric space, put

$$A \underset{C}{\downarrow} B$$

if and only if for all $a \in A, b \in B$ there is some $c \in C$ such that

$$d(a,b) = d(a,c) + d(c,b).$$

Put

$$A \downarrow B$$

if all $a \in A, b \in B$ are at *maximal possible* distance.

Proposition

Γ_{tr} *is ω-stable, 1-ample, but not 2-ample.*

Furthermore, every witness to 1-ampleness is (essentially) given by two neighbouring vertices.

Projective space $\mathbb{P}^{k+1}(K)$ as graph

For $0 \leq i \leq k$ let i-vertices $=$ $(i+1)$-dimensional subspaces

edges given by $\subseteq$ (symmetrized)

For a maximal flag $U_0, \ldots, U_k$ consisting of subspaces of $\mathbb{P}^{k+1}(K)$ we see by metric independence in the reduced graph that

$$U_0, \ldots U_i \not\perp U_{i+1} \ldots U_k \quad \text{and} \quad U_0, \ldots U_{i-1} \underset{U_i}{\perp} U_{i+1} \ldots U_k$$

Use action of GL_{k+2} on $\mathbb{P}^{k+1}(K)$ to show:

$$\mathrm{acl}(U_0 \ldots U_{i-1} U_i) \cap \mathrm{acl}(U_0 \ldots U_{i-1} U_{i+1}) = \mathrm{acl}(U_0 \ldots U_{i-1})$$

Thus, projective $k+1$-space (as a coloured graph) is k-ample.

Projective spaces are ample

But by the main theorem of projective geometry, the field is definable in the coloured graph as above. Hence, $\mathbb{P}^{k+1}(K)$ is n-ample for all n.

Question

Does every 2-ample strongly minimal structure define an infinite field?

How to construct possible counterexamples?

Morley rank

Recall the definition of *Morley rank*:

Definition

Let M be a (saturated) L-structure, X a definable subset (of M^n).

- $MR(X) \geq 0$ if $X \neq \emptyset$;
- $MR(X) \geq \alpha + 1$ if there are disjoint definable set $X_i \subset X, i < \omega$ such that $MR(X_i) \geq \alpha$;
- $MR(X) \geq \lambda$ for limit ordinal λ if $MR(X) \geq \alpha$ for all $\alpha < \lambda$.

Put $MR(X) = \alpha$ if $MR(X) \geq \alpha$ and $MR(X) \not\geq \alpha + 1$.

Trees and buildings

For a vertex $a \in \Gamma_{tr}$ or $a \in \mathbb{P}^3(K)$, the set of neighbours of a is strongly minimal.

Furthermore, the definable set $\{x \in \Gamma_{tr} \colon d(x,a) = s\}$ has Morley rank s. In particular, $MR(\Gamma_{tr}) = \omega$.

Similarly, the set $\{x \in \mathbb{P}^3(K) \colon d(x,a) = 2\}$ has Morley rank 2 and *so* $MR(\mathbb{P}^3(K)) = 2$.

Trees and buildings

Now consider Γ_{tr} as a bipartite graph without cycles of infinite diameter: a geometry of type $\bullet \overset{\infty}{-} \bullet$

Similarly, $\mathbb{P}^3(K)$ as a simplicial complex is a bipartite graph by definition: Points, lines, diameter $= 3$, girth $= 6$, i.e. a geometry of type $\bullet \overset{3}{-} \bullet$

Note that this is the Dynkin diagram of $GL_3(K)$.

Projective space of dimension $k+1$ corresponds to the Dynkin diagram

$$\bullet \overset{3}{-} \bullet \overset{3}{-} \bullet \ldots \bullet \overset{3}{-} \bullet$$

Construct an analog of projective space, consisting of trees instead of triangles, a geometry of type

$$\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \ldots \bullet \overset{\infty}{-} \bullet$$

Right angled-buildings

Define inductively a geometry of rank $k+1$ and type

$$\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \ldots \bullet \overset{\infty}{-} \bullet$$

A geometry of type $\bullet \overset{\infty}{-} \bullet$ (and rank 2) is a tree with infinite valencies.

If rank k has been defined, define a geometry of rank $k+1$ and type

$$\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \ldots \bullet \overset{\infty}{-} \bullet$$

as a geometry with $k+1$ types of vertices such that for all vertices x of type 0 and $k+1$ the *residues* (i.e. the set of vertices incident with x) are geometries of rank k and type

$$\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \ldots \bullet \overset{\infty}{-} \bullet$$

These geometries are *right-angled buildings.*

Right angled-buildings

Theorem (Tent, Baudisch-Pizarro-Ziegler)

The right-angled buildings of dimension $k+1$ and type

$$\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \ldots \bullet \overset{\infty}{-} \bullet$$

are ω-stable, k-ample, and not $k+1$-ample.
In particular, they do not define any infinite field (nor any infinite group).

Furthermore, any witness to k-ampleness arises (essentially) from a maximal flag.

Clearly, these geometries have infinite Morley rank. In order to obtain ample strongly minimal structures, we have to bound the diameter of the geometries.....

Geometries of type $\bullet \overset{n}{—} \bullet \overset{n}{—} \bullet \ldots \bullet \overset{n}{—} \bullet$

We already constructed such geometries in dimension 2:

Theorem (Tent, 2000)

For all $n \geq 3$ there exist strongly minimal structures that define geometries of type of type $\bullet \overset{n}{—} \bullet$.

Using these we obtain desired structures:

Theorem (Ammer-Tent)

For all $k \geq 1$ there exist strongly minimal structures that are k-ample, but not $k+1$-ample.

We construct almost strongly minimal geometries of type

$$\bullet \overset{n}{—} \bullet \overset{n}{—} \bullet \ldots \bullet \overset{n}{—} \bullet .$$

More on Morley rank

How to build a strongly minimal structure, or, more generally, a structure of finite Morley rank *from scratch*?

Definition

Suppose M is (saturated) L-structure, $\bar{a} \in M^n$, $A \subset M$. Define

$$MR(\bar{a}/A) = \min\{MR(X) \colon X \subset M^n, a \in X,\ X \quad L(A)\text{-definable}\}$$

Note that $MR(\bar{a}/A) = 0$ if and only if $a \in \mathrm{acl}(A)$.

Clearly $MR(\bar{a}/A) \geq MR(\bar{a}/AB)$. In fact, we have

$$\bar{a} \underset{A}{\downarrow} B \quad \text{if and only if} \quad MR(\bar{a}/A) = MR(\bar{a}/AB).$$

So $\bar{a}$ is independent from B over A if B does not add any information about $\bar{a}$ that wasn't already known from A.

Automorphisms

Want to construct a structure with built-in Morley rank. We saw before:

Remark

If $\alpha \in \mathrm{Aut}_A(M)$, then for all $x \in M$, the elements x and $\alpha(x)$ satisfy the same $L(A)$-formulas. In particular $MR(\bar{x}/A) = MR(\alpha(\bar{x})/A)$.

In order to use this observation, want to construct structures with many automorphisms

Fraïssé's construction

Theorem (Fraïssé)

Let $\mathcal{C}$ be a countable class of finitely generated structures, closed under

- *(AP) amalgamation and*
- *(JEP) joint embedding.*

Then there is a countable structure M which is

- *$\mathcal{C}$-universal, i.e. every structure $U \in \mathcal{C}$ can be embedded into M, and*
- *$\mathcal{C}$-homogeneous, i.e. if A, B are substructures of M, $f : A \longrightarrow B$ an isomorphism, and A, B are isomorphic to some structure $U \in \mathcal{C}$, then there is an automorphism of M extending f.*

Furthermore, M is unique up to isomorphism.

Fraïssé's method with Hrushovski's twist

Theorem (Fraïssé-Hrushovski)

Let $(\mathcal{C}, \leq)$ be a countable class of finitely generated structures with a partial order $\leq$, closed under

- *($\leq$-AP) $\leq$-amalgamation and*
- *($\leq$-JEP) $\leq$-joint embedding.*

Then there is a countable structure M which is

- *($\mathcal{C}$, $\leq$-)-universal, i.e. every structure $U \in \mathcal{C}$ can be $\leq$-embedded into M, and*
- *($\mathcal{C}$, $\leq$-)-homogeneous, i.e. if A, B are $\leq$-substructures of M, $f : A \longrightarrow B$ an isomorphism, and A, B are isomorphic to some structure $U \in \mathcal{C}$, then there is an automorphism of M extending f.*

Furthermore, M is unique up to isomorphism.

Hrushovski's method

How to choose the relation $\leq$ on a class of structures?

What we want:

$A \leq B$ if and only if B does not add information about A.

Introduce a function δ on the structures in $\mathcal{C}$. This function should eventually agree with the Morley rank.

Want δ to determine $\leq$ in the following way:

$$A \leq B \text{ if and only if for all } A \subset C \subset B \text{ we have } \delta(C) \geq \delta(A).$$

Or equivalently, $A \leq B$ if $\delta(C/A) \geq 0$ for all $A \subset C \subset B$.

In this case say that A is *strong* in B.

Example of construction

Definition

A generalized n-gon is a bipartite graph with diameter n and girth $2n$ with valencies at least 3.

- A generalized 2-gon is a complete bipartite graph.
- A generalized 3-gon is a projective plane.
- A generalized n-gon is a geometry of type $\bullet \overset{n}{-} \bullet$.

Theorem (T.)

For all $n \geq 3$ there exist generalized n-gons of Morley rank $n-1$.

Example of construction

Want to construct these generalized n-gons by amalgamating a class $\mathcal{C}$ of finite *partial n-gons*, i.e. finite bipartite graphs not containing any $2k$-cycles for $k < n$.

In the finaly generalized n-gons Γ, want for all vertices a the set $\{x \in \Gamma \colon d(x, a) = 1\}$ to be strongly minimal.

So have to define the function δ on $\mathcal{C}$ accordingly:

For a point-line pair (p, ℓ) we want $\delta(p/\ell) = \delta(\ell/p) = 1$.

Because the diameter is n (and the graph is bipartite) adding a path such that the distance between vertices is $n - 1$ or n must be a strong extension. Thus, we define for $A \in \mathcal{C}$:

$$\delta(A) = (n - 1)|A| - (n - 2)|E_A|.$$

Details of construction

Amalgamate $(\mathcal{C}, \leq)$ to obtain a generalized n-gon Γ.

However, in order to have δ describe the Morley rank on Γ, we have to ensure that for subsets A, B in Γ, there should only be finitely many copies of A over B if $\delta(A/B) = 0$.

Thus, we reduce $\mathcal{C}$ to a subclass $\mathcal{C}_\mu$ consisting only of those $U \in \mathcal{C}$ with $\delta(U) \geq 0$ and such that for any pair A, B with $\delta(A/B) = 0$, and any copy of B in U there are only a fixed number of copies of A over B inside.

Now one has to show that this class $(\mathcal{C}_\mu, \leq)$ satisfies the amalgamation property, i.e. we can amalgamate in such a way that the algebraicity condition is preserved.

The result of this process is a generalized n-gon Γ for which the set of points on each line is a strongly minimal set.

Generalized n-gons are 1-ample

Again we can show that for any $a \in \Gamma$ and $s < n$, the set $\{x \in \Gamma \colon d(x, a) = s\}$ has Morley rank s.

From this we conclude:

- independence is given by metric independence,
- any incident point-line pair is a witness for 1-ampleness,
- any witness for 1-ampleness is essentially an incident point-line pair, and
- Γ is not 2-ample.

Building-like geometries of higher rank

As with right-angled buildings we now construct higher rank analogs:

Theorem (Ammer-Tent)

For any $k \geq 1$ and any $n \geq 2 \cdot 5^{k-1} + 1$ there are almost strongly minimal geometries of type

$$\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet \cdots \bullet \overset{n}{-} \bullet .$$

These geometries are k-ample, but not $k+1$-ample.

The construction proceeds again by a variant of Hrushvoski's method, using induction on k and the generalized n-gons for $k = 1$.

Define δ_k inductively:

- put $\delta_1 = \delta$;
- put $\delta_k(x_0) = 5 \cdot \delta_{k-1}(x_0)$ and $\delta_k(x_k) = 5 \cdot \delta_{k-1}(x_k)$,
- for $A \subset res_>(x_i)$, put $\delta_k(A/x_i \ldots x_0) = \delta_k(A/x_i) = \delta_{i-1}(A)$, similarly for $A \subset res_<(x_i)$.

This δ-function is not *submodular*, i.e. we can have

$$\delta(A/B) \geq \delta(A/BC).$$

This makes the construction difficult.

We show that if $A \leq B$, then every element from $B \setminus A$ has a *gate* to A and

$$\delta(B/A) = \delta(B/gate(B)).$$

This is sufficient to obtain a version of submodularity that we can work with.

Conclusion: For all $k \geq 1$ there exist strongly minimal structures that are k-ample and not $k+1$-ample.