

# Model theoretic ampleness

Katrin Tent  
Westfälische Wilhelms-Universität Münster

Udine, July 2018

## Explain ‘ampleness’ and its connection to projective spaces

### 1. Background and definitions

Strongly minimal structures, dimension, modularity, independence

### 2. Stability and ampleness

Free groups, simplicial complexes, projective spaces

Today:

### 3. Recent examples of ample structures without fields

Constructions of ample structures by Hrushovski method

Recall

## Definition (Pillay-Evans)

A *stable* structure  $M$  is  $k$ -ample for some  $k \geq 1$  if there are tuples  $\bar{a}_0, \dots, \bar{a}_k \subset M$  such that (possibly after naming parameters) for all  $0 \leq i < k$  the following hold:

- $\bar{a}_0 \dots \bar{a}_i \not\perp \bar{a}_{i+1} \dots \bar{a}_k$ ;
- $\bar{a}_0 \dots \bar{a}_{i-1} \perp_{\bar{a}_i} \bar{a}_{i+1} \dots \bar{a}_k$ ;
- $\text{acl}(\bar{a}_0 \dots \bar{a}_{i-1} \bar{a}_i) \cap \text{acl}(\bar{a}_0 \dots \bar{a}_{i-1} \bar{a}_{i+1}) = \text{acl}(\bar{a}_0 \dots \bar{a}_{i-1})$ .

# Independence

For finite sets  $A, B, C$  contained in some metric space, put

$$A \downarrow_C B$$

if and only if for all  $a \in A, b \in B$  there is some  $c \in C$  such that

$$d(a, b) = d(a, c) + d(c, b).$$

Put

$$A \downarrow B$$

if all  $a \in A, b \in B$  are at *maximal possible* distance.

## Proposition

$\Gamma_{tr}$  is  $\omega$ -stable, 1-ample, but not 2-ample.

Furthermore, every witness to 1-ampleness is (essentially) given by two neighbouring vertices.

# Projective space $\mathbb{P}^{k+1}(K)$ as graph

For  $0 \leq i \leq k$  let  $i$ -vertices =  $(i + 1)$ -dimensional subspaces

edges given by  $\subseteq$  (symmetrized)

For a maximal flag  $U_0, \dots, U_k$  consisting of subspaces of  $\mathbb{P}^{k+1}(K)$  we see by metric independence in the reduced graph that

$$U_0, \dots, U_i \not\perp U_{i+1} \dots U_k \quad \text{and} \quad U_0, \dots, U_{i-1} \underset{U_i}{\perp} U_{i+1} \dots U_k$$

Use action of  $GL_{k+2}$  on  $\mathbb{P}^{k+1}(K)$  to show:

$$\text{acl}(U_0 \dots U_{i-1} U_i) \cap \text{acl}(U_0 \dots U_{i-1} U_{i+1}) = \text{acl}(U_0 \dots U_{i-1})$$

Thus, projective  $k + 1$ -space (as a coloured graph) is  $k$ -ample.

# Projective spaces are ample

But by the main theorem of projective geometry, the field is definable in the coloured graph as above. Hence,  $\mathbb{P}^{k+1}(K)$  is  $n$ -ample for all  $n$ .

## Question

*Does every 2-ample strongly minimal structure define an infinite field?*

How to construct possible counterexamples?

Recall the definition of *Morley rank*:

## Definition

Let  $M$  be a (saturated)  $L$ -structure,  $X$  a definable subset (of  $M^n$ ).

- $MR(X) \geq 0$  if  $X \neq \emptyset$ ;
- $MR(X) \geq \alpha + 1$  if there are disjoint definable set  $X_i \subset X, i < \omega$  such that  $MR(X_i) \geq \alpha$ ;
- $MR(X) \geq \lambda$  for limit ordinal  $\lambda$  if  $MR(X) \geq \alpha$  for all  $\alpha < \lambda$ .

Put  $MR(X) = \alpha$  if  $MR(X) \geq \alpha$  and  $MR(X) \not\geq \alpha + 1$ .

For a vertex  $a \in \Gamma_{tr}$  or  $a \in \mathbb{P}^3(K)$ , the set of neighbours of  $a$  is strongly minimal.

Furthermore, the definable set  $\{x \in \Gamma_{tr} : d(x, a) = s\}$  has Morley rank  $s$ . In particular,  $MR(\Gamma_{tr}) = \omega$ .

Similarly, the set  $\{x \in \mathbb{P}^3(K) : d(x, a) = 2\}$  has Morley rank 2 and so  $MR(\mathbb{P}^3(K)) = 2$ .



# Trees and buildings

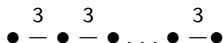
Now consider  $\Gamma_{tr}$  as a bipartite graph without cycles of infinite diameter: a geometry of type  $\bullet - \overset{\infty}{\bullet}$

Similarly,  $\mathbb{P}^3(K)$  as a simplicial complex is a bipartite graph by definition:

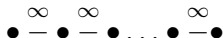
Points, lines, diameter = 3, girth = 6, i.e. a geometry of type  $\bullet - \overset{3}{\bullet}$

Note that this is the Dynkin diagram of  $GL_3(K)$ .

Projective space of dimension  $k + 1$  corresponds to the Dynkin diagram

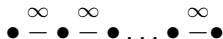


Construct an analog of projective space, consisting of trees instead of triangles, a geometry of type



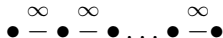
# Right angled-buildings

Define inductively a geometry of rank  $k + 1$  and type

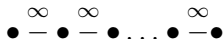


A geometry of type  $\bullet - \overset{\infty}{\bullet}$  (and rank 2) is a tree with infinite valencies.

If rank  $k$  has been defined, define a geometry of rank  $k + 1$  and type



as a geometry with  $k + 1$  types of vertices such that for all vertices  $x$  of type 0 and  $k + 1$  the *residues* (i.e. the set of vertices incident with  $x$ ) are geometries of rank  $k$  and type



These geometries are *right-angled buildings*.

## Theorem (Tent, Baudisch-Pizarro-Ziegler)

*The right-angled buildings of dimension  $k + 1$  and type*

$$\bullet - \overset{\infty}{\bullet} - \overset{\infty}{\bullet} - \dots - \overset{\infty}{\bullet} - \bullet$$

*are  $\omega$ -stable,  $k$ -ample, and not  $k + 1$ -ample.*

*In particular, they do not define any infinite field (nor any infinite group).*

*Furthermore, any witness to  $k$ -ampleness arises (essentially) from a maximal flag.*

Clearly, these geometries have infinite Morley rank. In order to obtain ample strongly minimal structures, we have to bound the diameter of the geometries.....

# Geometries of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet \dots \bullet \overset{n}{-} \bullet$

We already constructed such geometries in dimension 2:

## Theorem (Tent, 2000)

*For all  $n \geq 3$  there exist strongly minimal structures that define geometries of type of type  $\bullet \overset{n}{-} \bullet$ .*

Using these we obtain desired structures:

## Theorem (Ammer-Tent)

*For all  $k \geq 1$  there exist strongly minimal structures that are  $k$ -ample, but not  $k + 1$ -ample.*

We construct almost strongly minimal geometries of type

$$\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet \dots \bullet \overset{n}{-} \bullet .$$

## More on Morley rank

How to build a strongly minimal structure, or, more generally, a structure of finite Morley rank *from scratch*?

### Definition

Suppose  $M$  is (saturated)  $L$ -structure,  $\bar{a} \in M^n$ ,  $A \subset M$ . Define

$$MR(\bar{a}/A) = \min\{MR(X) : X \subset M^n, a \in X, X \text{ } L(A)\text{-definable}\}$$

Note that  $MR(\bar{a}/A) = 0$  if and only if  $a \in \text{acl}(A)$ .

Clearly  $MR(\bar{a}/A) \geq MR(\bar{a}/AB)$ . In fact, we have

$$\bar{a} \underset{A}{\perp} B \quad \text{if and only if} \quad MR(\bar{a}/A) = MR(\bar{a}/AB).$$

So  $\bar{a}$  is independent from  $B$  over  $A$  if  $B$  does not add any information about  $\bar{a}$  that wasn't already known from  $A$ .

Want to construct a structure with built-in Morley rank. We saw before:

## Remark

*If  $\alpha \in \text{Aut}_A(M)$ , then for all  $x \in M$ , the elements  $x$  and  $\alpha(x)$  satisfy the same  $L(A)$ -formulas. In particular  $MR(\bar{x}/A) = MR(\alpha(\bar{x})/A)$ .*

In order to use this observation, want to construct structures with many automorphisms

## Theorem (Fraïssé)

Let  $\mathcal{C}$  be a countable class of finitely generated structures, closed under

- (AP) amalgamation and
- (JEP) joint embedding.

Then there is a countable structure  $M$  which is

- $\mathcal{C}$ -universal, i.e. every structure  $U \in \mathcal{C}$  can be embedded into  $M$ , and
- $\mathcal{C}$ -homogeneous, i.e. if  $A, B$  are substructures of  $M$ ,  $f : A \rightarrow B$  an isomorphism, and  $A, B$  are isomorphic to some structure  $U \in \mathcal{C}$ , then there is an automorphism of  $M$  extending  $f$ .

Furthermore,  $M$  is unique up to isomorphism.

# Fraïssé's method with Hrushovski's twist

## Theorem (Fraïssé-Hrushovski)

Let  $(\mathcal{C}, \leq)$  be a countable class of finitely generated structures with a partial order  $\leq$ , closed under

- $(\leq\text{-AP})$   $\leq$ -amalgamation and
- $(\leq\text{-JEP})$   $\leq$ -joint embedding.

Then there is a countable structure  $M$  which is

- $(\mathcal{C}, \leq\text{-})$ -universal, i.e. every structure  $U \in \mathcal{C}$  can be  $\leq$ -embedded into  $M$ , and
- $(\mathcal{C}, \leq\text{-})$ -homogeneous, i.e. if  $A, B$  are  $\leq$ -substructures of  $M$ ,  $f : A \rightarrow B$  an isomorphism, and  $A, B$  are isomorphic to some structure  $U \in \mathcal{C}$ , then there is an automorphism of  $M$  extending  $f$ .

Furthermore,  $M$  is unique up to isomorphism.



# Hrushovski's method

How to choose the relation  $\leq$  on a class of structures?

What we want:

$A \leq B$  if and only if  $B$  does not add information about  $A$ .

Introduce a function  $\delta$  on the structures in  $\mathcal{C}$ . This function should eventually agree with the Morley rank.

Want  $\delta$  to determine  $\leq$  in the following way:

$A \leq B$  if and only if for all  $A \subset C \subset B$  we have  $\delta(C) \geq \delta(A)$ .

Or equivalently,  $A \leq B$  if  $\delta(C/A) \geq 0$  for all  $A \subset C \subset B$ .

In this case say that  $A$  is *strong* in  $B$ .

# Example of construction

## Definition

A generalized  $n$ -gon is a bipartite graph with diameter  $n$  and girth  $2n$  with valencies at least 3.

- A generalized 2-gon is a complete bipartite graph.
- A generalized 3-gon is a projective plane.
- A generalized  $n$ -gon is a geometry of type  $\bullet \overset{n}{-} \bullet$ .

## Theorem (T.)

*For all  $n \geq 3$  there exist generalized  $n$ -gons of Morley rank  $n - 1$ .*

## Example of construction

Want to construct these generalized  $n$ -gons by amalgamating a class  $\mathcal{C}$  of finite *partial  $n$ -gons*, i.e. finite bipartite graphs not containing any  $2k$ -cycles for  $k < n$ .

In the finally generalized  $n$ -gons  $\Gamma$ , want for all vertices  $a$  the set  $\{x \in \Gamma : d(x, a) = 1\}$  to be strongly minimal.

So have to define the function  $\delta$  on  $\mathcal{C}$  accordingly:

For a point-line pair  $(p, \ell)$  we want  $\delta(p/\ell) = \delta(\ell/p) = 1$ .

Because the diameter is  $n$  (and the graph is bipartite) adding a path such that the distance between vertices is  $n - 1$  or  $n$  must be a strong extension. Thus, we define for  $A \in \mathcal{C}$ :

$$\delta(A) = (n - 1)|A| - (n - 2)|E_A|.$$

## Details of construction

Amalgamate  $(\mathcal{C}, \leq)$  to obtain a generalized  $n$ -gon  $\Gamma$ .

However, in order to have  $\delta$  describe the Morley rank on  $\Gamma$ , we have to ensure that for subsets  $A, B$  in  $\Gamma$ , there should only be finitely many copies of  $A$  over  $B$  if  $\delta(A/B) = 0$ .

Thus, we reduce  $\mathcal{C}$  to a subclass  $\mathcal{C}_\mu$  consisting only of those  $U \in \mathcal{C}$  with  $\delta(U) \geq 0$  and such that for any pair  $A, B$  with  $\delta(A/B) = 0$ , and any copy of  $B$  in  $U$  there are only a fixed number of copies of  $A$  over  $B$  inside.

Now one has to show that this class  $(\mathcal{C}_\mu, \leq)$  satisfies the amalgamation property, i.e. we can amalgamate in such a way that the algebraicity condition is preserved.

The result of this process is a generalized  $n$ -gon  $\Gamma$  for which the set of points on each line is a strongly minimal set.

# Generalized $n$ -gons are 1-ample

Again we can show that for any  $a \in \Gamma$  and  $s < n$ , the set  $\{x \in \Gamma : d(x, a) = s\}$  has Morley rank  $s$ .

From this we conclude:

- independence is given by metric independence,
- any incident point-line pair is a witness for 1-amplessness,
- any witness for 1-amplessness is essentially an incident point-line pair, and
- $\Gamma$  is not 2-ample.

# Building-like geometries of higher rank

As with right-angled buildings we now construct higher rank analogs:

## Theorem (Ammer-Tent)

For any  $k \geq 1$  and any  $n \geq 2 \cdot 5^{k-1} + 1$  there are almost strongly minimal geometries of type

$$\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet \dots \bullet \overset{n}{-} \bullet .$$

*These geometries are  $k$ -ample, but not  $k + 1$ -ample.*

The construction proceeds again by a variant of Hrushovski's method, using induction on  $k$  and the generalized  $n$ -gons for  $k = 1$ .

Define  $\delta_k$  inductively:

- put  $\delta_1 = \delta$ ;
- put  $\delta_k(x_0) = 5 \cdot \delta_{k-1}(x_0)$  and  $\delta_k(x_k) = 5 \cdot \delta_{k-1}(x_k)$ ,
- for  $A \subset \text{res}_{>}(x_i)$ , put  $\delta_k(A/x_i \dots x_0) = \delta_k(A/x_i) = \delta_{i-1}(A)$ , similarly for  $A \subset \text{res}_{<}(x_i)$ .

This  $\delta$ -function is not *submodular*, i.e. we can have

$$\delta(A/B) \geq \delta(A/BC).$$

This makes the construction difficult.

We show that if  $A \leq B$ , then every element from  $B \setminus A$  has a *gate* to  $A$  and

$$\delta(B/A) = \delta(B/\text{gate}(B)).$$

This is sufficient to obtain a version of submodularity that we can work with.

**Conclusion:** For all  $k \geq 1$  there exist strongly minimal structures that are  $k$ -ample and not  $k + 1$ -ample.